

GENERATING FUNCTIONS FOR K-THEORETIC DONALDSON INVARIANTS AND LE POTIER'S STRANGE DUALITY

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ABSTRACT. For a projective algebraic surface X , with an ample line bundle H , let $M_H^X(c)$ be the moduli space H -semistable sheaves \mathcal{E} of class c in the Grothendieck group $K(X)$. We write $c = (r, c_1, c_2)$, or $c = (r, c_1, \chi)$ with r the rank, c_1, c_2 , the Chern classes and χ the holomorphic Euler characteristic. We also write $M_H^X(2, c_1, c_2) = M_X^X(c_1, d)$, with $d = 4c_2 - c_1^2$. The K -theoretic Donaldson invariants are the holomorphic Euler characteristics $\chi(M_H^X(c_1, d), \mu(L))$, where $\mu(L)$ is the determinant line bundle associated to a line bundle on X . More generally for suitable classes $c^* \in K(X)$ there is a determinant line bundle \mathcal{D}_{c, c^*} on $M_H^X(c)$. We first compute some generating functions for K -theoretic Donaldson invariants on \mathbb{P}^2 and rational ruled surfaces, using the wallcrossing formula of [11].

Then we show that Le Potier's strange duality conjecture relating $H^0(M_H^X(c), \mathcal{D}_{c, c^*})$ and $H^0(M_H^X(c^*), \mathcal{D}_{c^*, c})$ holds for the cases $c = (2, c_1 = 0, c_2 > 2)$ and $c^* = (0, L, \chi = 0)$ with $L = -K_X$ on \mathbb{P}^2 , and $L = -K_X$ or $-K_X + F$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and $\widehat{\mathbb{P}}^2$ with F the fiber class of the ruling, and also the case $c = (2, H, c_2)$ and $c^* = (0, 2H, \chi = -1)$ on \mathbb{P}^2 .

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1. INTRODUCTION

In this whole paper let X be a simply connected nonsingular projective surface over \mathbb{C} , with its anti-canonical divisor $-K_X$ ample. In particular X is a rational surface. For H ample on X , $c_1 \in H^2(X, \mathbb{Z})$, $c_2 \in \mathbb{Z}$, let $M := M_H^X(c_1, d)$ with $d = 4c_2 - c_1^2$ be the moduli space of H -semistable sheaves with Chern classes c_1, c_2 . For a line bundle L on X there exists a corresponding determinant line bundle $\mu(L)$ on M . The Donaldson invariants are given as the top self intersection numbers $\int_M c_1(\mu(L))^{\dim M}$ on the moduli spaces. In [11] the K -theoretic Donaldson invariants are introduced as the holomorphic Euler characteristics $\chi(M, \mu(L))$. Like for the usual Donaldson invariants it is an interesting problem to understand the K -theoretic Donaldson invariants and determine their generating functions. In this paper we will do this in a number of cases for \mathbb{P}^2 and rational ruled surfaces. This result is then applied to prove some cases of Le Potier's strange duality conjecture, which is a duality between spaces of sections of determinant bundles on different moduli spaces of sheaves on surfaces.

1.1. K -theoretic Donaldson invariants. Let $K(X)$ be the Grothendieck group of coherent sheaves over X . Let c be an element in $K(X)$, which is the class of a coherent rank 2 sheaf with Chern classes c_1, c_2 . We write $M_H^X(c) = M_H^X(c_1, d)$ with $d = 4c_2 - c_1^2$ for the moduli space of H -semistable sheaves in class c . Let L be a line bundle on X and assume that $\langle c_1(L), c_1 \rangle$ is even with $\langle -, - \rangle$ the intersection form on $H^2(X, \mathbb{Z})$. Then we put

$$(1.1) \quad v(L) := (1 - L^{-1}) + \left\langle \frac{c_1(L)}{2}, (c_1(L) + K_X - c_1) \right\rangle [\mathcal{O}_x].$$

Note that $v(L)$ is independent of c_2 . Assume that H is c -general. Then we have a well-defined *determinant line bundle* $\mu(L) := \lambda(v(L)) \in \text{Pic}(M_H^X(c_1, d))$ associated to $v(L)$. The *K-theoretic Donaldson invariant* of X with respect to L, c_1, d, H is $\chi(M_H^X(c_1, d), \mu(L))$.

1.2. Results for rational ruled surfaces. We denote by $\Sigma_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e))$ the e -th rational ruled surface. We will restrict to the cases $e = 0$, i.e. $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $e = 1$, i.e. $X = \widehat{\mathbb{P}^2}$, the blowup of \mathbb{P}^2 in a point. In the case $X = \mathbb{P}^1 \times \mathbb{P}^1$ we denote G the class of the fibre of the second projection to \mathbb{P}^1 . In the case $X = \widehat{\mathbb{P}^2}$ let H be the pullback of the hyperplane class on \mathbb{P}^2 and E the exceptional divisor. We write $F = H - E$ and $G = (H + E)/2$. Note that G only lies in $\frac{1}{2}H^2(\widehat{\mathbb{P}^2}, \mathbb{Z})$.

For power series $f(\Lambda) = \sum_{d \geq 0} f_d \Lambda^d$, $g(\Lambda) = \sum_{d \geq 0} g_d \Lambda^d \in \mathbb{Q}[[\Lambda]]$, we write $f(\Lambda) \equiv g(\Lambda)$ if there exists a $d_0 \geq 0$ with $f_d = g_d$ for all $d \geq d_0$.

Theorem 1.2. *For $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $X = \widehat{\mathbb{P}^2}$ and for $\frac{a}{b} \geq \frac{n+2}{4}$ the following hold.*

(1) *For $n \in \mathbb{Z}$ we have*

$$1 + \sum_{d > 0} \chi(M_{aF+bG}^X(F, d), \mu(nF)) \Lambda^d = \frac{1}{(1 - \Lambda^4)^{n+1}},$$

$$1 + (n+1)\Lambda^4 + \sum_{d > 4} \chi(M_{aF+bG}^X(0, d), \mu(nF)) \Lambda^d = \frac{1}{(1 - \Lambda^4)^{n+1}}.$$

(2) *For $n \in \mathbb{Z}$ in case $X = \mathbb{P}^1 \times \mathbb{P}^1$ and for $n \in \mathbb{Z} + \frac{1}{2}$ in case $X = \widehat{\mathbb{P}^2}$ we have*

$$1 + (2n+2)\Lambda^4 + \sum_{d > 4} \chi(M_{aF+bG}^X(0, d), \mu(nF + G)) \Lambda^d = \frac{1}{(1 - \Lambda^4)^{2n+2}}.$$

(3) *For $n \in \mathbb{Z}$ we have*

$$\sum_{d > 0} \chi(M_{aF+bG}^X(F, d), \mu(nF + 2G)) \Lambda^d = \frac{1}{2} \frac{(1 + \Lambda^4)^n - (1 - \Lambda^4)^n}{(1 - \Lambda^4)^{3n+3}},$$

$$1 + (3n+3)\Lambda^4 + \sum_{d > 4} \chi(M_{aF+bG}^X(0, d), \mu(nF + 2G)) \Lambda^d = \frac{1}{2} \frac{(1 + \Lambda^4)^n + (1 - \Lambda^4)^n}{(1 - \Lambda^4)^{3n+3}}.$$

(4) *The formulas of (1), (2), (3) above hold for all ample classes $aF + bG$ on X with $=$ replaced by \equiv .*

1.3. Results for the projective plane. Combining Theorem 1.2 and blowup formulas Lemma 2.3, Lemma 4.33 relating the invariants of a surface and its blowup in a point we get the following formulas for \mathbb{P}^2 .

Theorem 1.3.

$$\begin{aligned}
(1) \quad & 1 + 3\Lambda^4 + \sum_{d>4} \chi(M_H^{\mathbb{P}^2}(0, d), \mu(H))\Lambda^d = \frac{1}{(1 - \Lambda^4)^3}, \\
(2) \quad & 1 + 6\Lambda^4 + \sum_{d>4} \chi(M_H^{\mathbb{P}^2}(0, d), \mu(2H))\Lambda^d = \frac{1}{(1 - \Lambda^4)^6}, \\
(3) \quad & \sum_d \chi(M_H^{\mathbb{P}^2}(H, d), \mu(2H))\Lambda^d = \frac{\Lambda^3}{(1 - \Lambda^4)^6}, \\
(4) \quad & 1 + 10\Lambda^4 + \sum_{d>4} \chi(M_H^{\mathbb{P}^2}(0, d), \mu(3H))\Lambda^d = \frac{1 + \Lambda^8}{(1 - \Lambda^4)^{10}}.
\end{aligned}$$

1.4. Results on strange duality. We choose two elements $c, c^* \in K(X)$, such that both moduli spaces $M_H^X(c^*)$ and $M_H^X(c)$ are non-empty and the determinant line bundles $\lambda(c)$ and $\lambda(c^*)$ are well-defined over $M_H^X(c^*)$ and $M_H^X(c)$, respectively. Under suitable conditions, (see §2.5), there is a canonical map

$$SD_{c,c^*} : H^0(M_H^X(c), \lambda(c^*))^\vee \rightarrow H^0(M_H^X(c^*), \lambda(c)).$$

The strange duality conjecture says that SD_{c,c^*} should be an isomorphism.

The strange duality conjecture was first formulated for X a smooth curve in the 1990s (see [3] and [9]) and in this case been proved around 2007 (see [4], [18] and [5]). For X a surface, there does not exist until now a general formulation of the strange duality conjecture. There is a formulation for some special cases due to Le Potier (see [17] or [8]). We will prove the following cases of Le Potier's strange duality conjecture.

Theorem 1.4. *Let the polarization H be both c -general and c^* -general. Then the strange duality conjecture is true, i.e. the map SD_{c,c^*} is an isomorphism in the following three cases.*

- (1) $X = \mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or $\widehat{\mathbb{P}^2}$. $c = (2, 0, c_2)$ with $c_2 > 2$ and $c^* = (0, -K_X, \chi = 0)$, moreover if $X = \mathbb{P}^1 \times \mathbb{P}^1$ or $\widehat{\mathbb{P}^2}$, we chose the polarization of the form $H = aF + bG$ with $a \geq b$.
- (2) $X = \mathbb{P}^1 \times \mathbb{P}^1$ or $\widehat{\mathbb{P}^2}$ with $H = aF + bG$ and $\frac{a}{b} \geq \frac{5}{4}$, $c = (2, 0, c_2)$ with $c_2 > 2$ and $c^* = (0, 2G + 3F, \chi = 0)$.
- (3) $X = \mathbb{P}^2$ with H the hyperplane class, $c = (2, H, c_2)$ with $c_2 > 0$ and $c^* = (0, 2H, \chi = -1)$.

Actually we will show that essentially the strange duality conjecture holds for any polarization. But if H is not c -general, the formulation needs a slight modification (see Theorem 6.3 and Remark 6.33).

The strange duality for surfaces is a very interesting problem and many other people worked on it. For instance in the case \mathbb{P}^2 , Danila proves that Le Potier's strange duality holds for $c = (2, 0, c_2)$, $c^* = (0, dH, \chi = 0)$ with c_2 small and $d = 1, 2, 3$ ([7] and [8]); Abe shows that it holds for all $c = (2, 0, c_2)$, $c^* = (0, dH, \chi = 0)$ with $d = 1, 2$ ([1]); and the second author shows that it holds for all $c = (1, 0, c_2)$, $c^* = (0, dH, 0)$ (see Section 4.3 in [31]), and also for

all $c = (2, 0, c_2 = 2)$, $c^* = (0, dH, \chi = 0)$ ([32]). Marian and Oprea, and their collaborators have proven many results on the strange duality for K3 and abelian surfaces (e.g. [19], [20] and [21]). However, in general still very few results on this conjecture are known.

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2. BACKGROUND MATERIAL

For a class $\alpha \in H^*(X)$, denote $\langle \alpha \rangle := \int_X \alpha$. For $\alpha, \beta \in H^2(X)$ we write $\langle \alpha, \beta \rangle := \int_X \alpha \wedge \beta$ and $\beta^2 := \langle \beta, \beta \rangle$.

Let H be an ample divisor on X . A class c in the Grothendieck group $K(X)$ of coherent sheaves on X is determined by its rank and the Chern classes c_1, c_2 . Therefore we will also denote $c = (r, c_1, c_2)$ the class of rank r coherent sheaves on X with first and second Chern class c_1, c_2 . We also may write $c = (r, c_1, \chi)$ with χ standing for the holomorphic Euler characteristic. Let $M_H^X(c) = M_H^X(r, c_1, c_2)$ be the moduli space of H -semistable sheaves in class c , and let $M_H^X(c)^s$ be the open subset consisting of stable sheaves. If $r = 0$ and $c_1 = c_1(L)$ with L nontrivial and effective, then $M_H^X(r, c_1, c_2)$ is a moduli space of 1-dimensional semistable sheaves supported on curves in the linear system $|L|$. We will write $M_H^X(c_1, d)$ with $d := 4c_2 - c_1^2$ instead of $M_H^X(c)$ if $c = (2, c_1, c_2)$.

2.1. Determinant line bundles. We briefly review the determinant line bundles on the moduli space [10],[14], for more details we refer to [13, Chap. 8].

For a Noetherian scheme Y we denote by $K(Y)$ and $K^0(Y)$ the Grothendieck groups of coherent sheaves and locally free sheaves on Y respectively. If Y is nonsingular and quasiprojective, then $K(Y) = K^0(Y)$. In particular we have $K(X) = K^0(X)$ for the smooth projective surface X . If we want to distinguish a sheaf \mathcal{F} and its class in $K(Y)$, we denote the latter by $[\mathcal{F}]$, but we may also write \mathcal{F} for the class in $K(Y)$. For a proper morphism $f: Y_1 \rightarrow Y_2$ we have the pushforward homomorphism $f_!: K(Y_1) \rightarrow K(Y_2); [\mathcal{F}] \mapsto \sum_i (-1)^i [R^i f_* \mathcal{F}]$. For any morphism $f: Y_1 \rightarrow Y_2$ we have the pullback homomorphism $f^*: K^0(Y_2) \rightarrow K^0(Y_1); [\mathcal{F}] \mapsto [f^* \mathcal{F}]$ for a locally free sheaf \mathcal{F} on Y_2 . Let \mathcal{E} be a flat family of coherent sheaves of class c on X parametrized by a scheme S , then $\mathcal{E} \in K^0(X \times S)$. Let $p: X \times S \rightarrow S$, $q: X \times S \rightarrow X$ be the projections. Define $\lambda_{\mathcal{E}}: K(X) \rightarrow \text{Pic}(S)$ as the composition of the following homomorphisms:

$$(2.1) \quad K(X) = K^0(X) \xrightarrow{q^*} K^0(X \times S) \xrightarrow{[\mathcal{E}]} K^0(X \times S) \xrightarrow{p_!} K^0(S) \xrightarrow{\det^{-1}} \text{Pic}(S),$$

where q^* is the pull-back morphism, $[\mathcal{F}].[\mathcal{G}] := \sum_i (-1)^i [\text{Tor}_i(\mathcal{F}, \mathcal{G})]$ with $[\mathcal{F}]$ the class of sheaf \mathcal{F} in $K(X)$, and $p_!([\mathcal{F}]) = \sum_i (-1)^i [R^i p_* \mathcal{F}]$. Notice that $p_!([\mathcal{F}]) \in K^0(S)$ for \mathcal{F} S -flat by Proposition 2.1.10 in [13].

The following elementary facts are important for working with these line bundles:

- (1) $\lambda_{\mathcal{E}}$ is a homomorphism, i.e. $\lambda_{\mathcal{E}}(v_1 + v_2) = \lambda_{\mathcal{E}}(v_1) \otimes \lambda_{\mathcal{E}}(v_2)$.

(2) If $\mu \in \text{Pic}(S)$ is a line bundle, then $\lambda_{\mathcal{E} \otimes p^* \mu}(v) = \lambda_{\mathcal{E}}(v) \otimes \mu^{\chi(c \otimes v)}$.

(3) $\lambda_{\mathcal{E}}$ is compatible with base change: if $\phi : S' \rightarrow S$ is a morphism, then $\lambda_{\phi^* \mathcal{E}}(v) = \phi^* \lambda_{\mathcal{E}}(v)$.

However, in general there is no universal sheaf \mathcal{E} over $X \times M_H^X(c)$, and even if it exists, there is ambiguity caused by tensoring with the pull-back of a line bundle on $M_H^X(c)$. Define $K_c := c^\perp = \{v \in K(X) \mid \chi(v \otimes c) = 0\}$, and $K_{c,H} := c^\perp \cap \{1, h, h^2\}^{\perp\perp}$, where $h = [\mathcal{O}_H]$. Then we have a well-defined morphism $\lambda : K_c \rightarrow \text{Pic}(M_H^X(c)^s)$, and $\lambda : K_{c,H} \rightarrow \text{Pic}(M_H^X(c))$ satisfying the following properties:

- (1) The λ commute with the inclusions $K_{c,H} \subset K_c$ and $\text{Pic}(M_H^X(c)) \subset \text{Pic}(M_H^X(c)^s)$.
- (2) If \mathcal{E} is a flat family of semistable sheaves on X of class c parametrized by S , then we have $\phi_{\mathcal{E}}^*(\lambda(v)) = \lambda_{\mathcal{E}}(v)$ for all $v \in K_{c,H}$ with $\phi_{\mathcal{E}} : S \rightarrow M_H^X(c)$ the classifying morphism.
- (3) If \mathcal{E} is a flat family of stable sheaves, the same statement to (2) holds with $K_{c,H}$, $M_H^X(c)$ replaced by K_c , $M_H^X(c)^s$.

Since X is a simply connected surface, both the moduli space $M_H^X(c)$ and the determinant line bundle $\lambda(c^*)$ only depend on the images of c and c^* in $K(X)_{\text{num}}$. Here $K(X)_{\text{num}}$ is the Grothendieck group modulo numerical equivalence. We say that $u, v \in K(X)$ are numerically equivalent if $u - v$ is in the radical of the quadratic form $(u, v) \mapsto \chi(X, u \otimes v) \equiv \chi(u \otimes v)$.

Often $\lambda : K_{c,H} \rightarrow \text{Pic}(M_H^X(c))$ can be extended. For instance let $c = (2, c_1, c_2)$, then $\lambda(v(L))$ is well-defined over $M_H^X(c)$ if $\langle L, \xi \rangle = 0$ for all ξ a class of type (c_1, d) (see §2.2) with $\langle H, \xi \rangle = 0$. This can be seen easily from the construction of $\lambda(v(L))$ (e.g. see the proof of Theorem 8.1.5 in [13]), we will also explain more in details in Remark 6.1 in §6.

2.2. Walls. Denote by \mathcal{C} the ample cone of X . Then \mathcal{C} has a chamber structure: For a class $\xi \in H^2(X, \mathbb{Z}) \setminus \{0\}$, let $W^\xi := \{x \in \mathcal{C} \mid \langle x, \xi \rangle = 0\}$. Assume $W^\xi \neq \emptyset$. Let $c_1 \in \text{Pic}(X)$, $d \in \mathbb{Z}$ and $d \equiv -c_1^2 \pmod{4}$. Then we call ξ a *class of type* (c_1, d) and call W^ξ a *wall of type* (c_1, d) if the following conditions hold

- (1) $\xi + c_1$ is divisible by 2 in $H^2(X, \mathbb{Z})$,
- (2) $d + \xi^2 \geq 0$.

We call ξ a *class of type* c_1 , if $\xi + c_1$ is divisible by 2 in $H^2(X, \mathbb{Z})$. We say that $H \in \mathcal{C}$ lies on the wall W^ξ if $H \in W^\xi$. The *chambers of type* (c_1, d) are the connected components of the complement of the walls of type (c_1, d) in \mathcal{C} . Then $M_H^X(c_1, d)$ depends only on the chamber of type (c_1, d) of H .

Let $c \in K(X)$ be the class of an sheaf $\mathcal{F} \in M_H^X(c_1, d)$. We call H *general* with respect to c if all the strictly semistable sheaves in $M_H^X(c)$ are strictly semistable with respect to all ample divisors on X in a neighbourhood of H . It is easy to see that H is general with respect to c if and only if H does not lie on a wall W_ξ of type (c_1, d) such that $(d + \xi^2)/8 \in \mathbb{Z}_{\geq 0}$.

2.3. K-theoretic Donaldson invariants. Let L be a line bundle on X and assume that $\langle c_1(L), c_1 \rangle$ is even. Then for $c = (2, c_1, c_2)$, we put

$$(2.2) \quad v(L) := (1 - L^{-1}) + \left\langle \frac{c_1(L)}{2}, (c_1(L) + K_X - c_1) \right\rangle [\mathcal{O}_x] \in K_c.$$

Note that $v(L)$ is independent of c_2 . Assume that H is general with respect to c . Then we denote $\mu(L) := \lambda(v(L)) \in \text{Pic}(M_H^X(c_1, d))$. The *K-theoretic Donaldson invariant* of X , with respect to L, c_1, d, H is $\chi(M_H^X(c_1, d), \mu(L))$.

We recall the following blowup relation for the *K-theoretic* Donaldson invariants from [11, Cor. 1.8]. Let (X, H) be a polarized rational surface. Let \widehat{X} be the blowup of X in a point and E the exceptional divisor. In the following we always denote a class in $H^*(X, \mathbb{Z})$ and its pullback by the same letter. Let Q be an open subset of a suitable quot-scheme such that $M_H^X(c_1, d) = Q/GL(N)$. Assume that Q is smooth (e.g. $\langle -K_X, H \rangle > 0$). We choose $\epsilon > 0$ sufficiently small so that $H - \epsilon E$ is ample on \widehat{X} and there is no class ξ of type (c_1, d) or of type $(c_1 + E, d + 1)$ on \widehat{X} with $\langle \xi, H \rangle < 0 < \langle \xi, (H - \epsilon E) \rangle$. In case $c_1 = 0$ assume $d > 4$.

Lemma 2.3. *We have*

$$\begin{aligned}\chi(M_{H-\epsilon E}^{\widehat{X}}(c_1, d), \mu(L)) &= \chi(M_H(c_1, d), \mu(L)), \\ \chi(M_{H-\epsilon E}^{\widehat{X}}(c_1 + E, d + 1), \mu(L)) &= \chi(M_H(c_1, d), \mu(L))\end{aligned}$$

for any line bundle L on X such that $\langle L, c_1 \rangle$ is even and $\langle L, \xi \rangle = 0$ for ξ any class of type (c_1, d) on \widehat{X} with $\langle H, \xi \rangle = 0$.

Remark 2.4. If H is a general polarization, then $\mu(2K_X)$ is a line bundle on $M_H^X(c)$ which coincides with the dualizing sheaf on the locus of stable sheaves $M_H^X(c)^s$. If $\dim(M_H^X(c) \setminus M_H^X(c)^s) \leq \dim M_H^X(c) - 2$, then $\omega_{M_H^X(c)} = \mu(2K_X)$.

We introduce the generating function of the *K-theoretic* Donaldson invariants.

Definition 2.5. Let $c_1 \in H^2(X, \mathbb{Z})$. Let H be ample on X not on a wall of type (c_1) .

(1) If $c_1 \notin 2H^2(X, \mathbb{Z})$, let

$$(2.6) \quad \chi_{c_1}^{X,H}(L) := \sum_{d>0} \chi(M_H^X(c_1, d), \mu(L)) \Lambda^d.$$

(2) In case $c_1 = 0$ let \widehat{X} be the blowup of X in a point. Let E be the exceptional divisor. Let $\epsilon > 0$ be sufficiently small so that there is no class ξ of type $(E, d + 1)$ on \widehat{X} with $\langle \xi, H \rangle < 0 < \langle \xi, (H - \epsilon E) \rangle$. We put

$$(2.7) \quad \chi_0^{X,H}(L) := \sum_{d>4} \chi(M_H^X(0, d), \mu(L)) \Lambda^d + \left(\chi(M_{H-\epsilon E}^{\widehat{X}}(E, 5), \mu(L)) + \langle L, K_X \rangle - \frac{K_X^2 + L^2}{2} - 1 \right) \Lambda^4.$$

Remark 2.8. (1) Note that with this definition we have $\text{Coeff}_{\Lambda^d} [\chi_0^{X,H}(L)] = \chi(M_H^X(0, d), \mu(L))$ only for $d > 4$.

(2) The coefficient of Λ^4 of $\chi_0^{X,H}(L)$ has been chosen to make the generating functions $\chi_{c_1}^{X,H}(L)$ more compatible among each other. In particular it will lead to a cleaner blowup formula for $\chi_0^{X,H}(L)$.

2.4. Vanishing of higher cohomology. In this paper we will compute the holomorphic Euler characteristics $\chi(M_H^X(c_1, d), \mu(L))$ on rational surfaces. We then want to apply this to prove cases of Le Potier's strange duality, which is a statement about spaces of sections $H^0(M_H^X(c_1, d), \mu(L))$. Thus we need to see that in the cases considered $\chi(M_H^X(c_1, d), \mu(L)) = \dim H^0(M_H^X(c_1, d), \mu(L))$. We will show this using arguments closely related to [11, §1.4] Let L be a numerically effective line bundle on X .

Proposition 2.9. *Fix c_1, d . Let H be an ample line bundle on X which is general with respect to (c_1, d) . If c_1 is not divisible by 2 in $H^2(X, \mathbb{Z})$ or $d > 8$, we have $H^i(M_H^X(c_1, d), \mu(L)) = 0$ for all $i > 0$, in particular*

$$\dim H^0(M_H^X(c_1, d), \mu(L)) = \chi(M_H^X(c_1, d), \mu(L)).$$

Proof. As $-K_X$ is ample on X , and L is numerically effective, we get that $L - 2K_X$ is ample on X . By [13, Prop.8.3.2.] there exists a positive integer n , such that $\mu(L - 2K_X)^{\otimes n}$ is globally generated on $M_H^X(c_1, d)$. Denote by ω_M the dualizing sheaf of $M_H^X(c_1, d)$. As $-K_X$ is ample we have by [6] that $M_H^X(c_1, d)$ is normal and has only rational singularities. Therefore [29, Cor.7.70] gives that $H^i(M_H^X(c_1, d), \mu(L - 2K_X) \otimes \omega_M) = 0$ for $i > 0$. If c_1 is not divisible by 2 in $H^2(X, \mathbb{Z})$, then, by our assumption that H is general, the moduli space $M_H^X(c_1, d)$ consists only of stable sheaves. If $c_1 = 0$, again using that H is general, we see that the strictly semistable points of $M_H^X(c_1, d)$ are of the form $\mathcal{I}_Z(c_1/2) \oplus \mathcal{I}_W(c_1/2)$ for 0-dimensional subschemes Z, W of X of length $d/8$. In particular if d is not divisible by 8, $M_H^X(c_1, d)$ consists only of stable sheaves, and if d is divisible by 8, the dimension of the locus M^{sss} of strictly semistable sheaves is $d/2$. On the other hand $M_H^X(c_1, d)$ has pure dimension $d - 3$. Thus if $d > 8$ we get M^{sss} has codimension at least 2 in $M_H^X(c_1, d)$. In all these cases Remark 2.4 says that $\omega_M = \mu(2K_X)$. Thus by the above $H^i(M_H^X(c_1, d), \mu(L - 2K_X) \otimes \omega_M) = H^i(M_H^X(c_1, d), \mu(L)) = 0$ for $i > 0$. \square

2.5. Strange duality. We briefly review the strange duality conjecture from [17]. Let $c, c^* \in K(X)_{num}$ with $c \in K_{c^*}$. Let H be ample line bundle on X which is both c -general and c^* -general. Write $\mathcal{D}_{c,c^*} := \lambda(c^*) \in \text{Pic}(M_H^X(c))$, $\mathcal{D}_{c^*,c} := \lambda(c) \in \text{Pic}(M_H^X(c^*))$. Assume that all H -semistable sheaves \mathcal{F} on X of class c and all H -semistable sheaves \mathcal{G} on X of class c^* satisfy

- (1) $\underline{\text{Tor}}_i(\mathcal{F}, \mathcal{G}) = 0$ for all $i \geq 1$,
- (2) $H^2(X, \mathcal{F} \otimes \mathcal{G}) = 0$.

Both conditions are automatically satisfied if c is not of dimension 0 and c^* is of dimension 1 (see [17, p.9]). If $c = (2, c_1, c_2)$ and $c^* = (0, L, \chi = -\langle \frac{c_1(L)}{2} \cdot c_1 \rangle)$, then $\mathcal{D}_{c,c^*} = \mu(L)$.

Put $\mathcal{D} := \mathcal{D}_{c,c^*} \boxtimes \mathcal{D}_{c^*,c} \in \text{Pic}(M_H^X(c) \times M_H^X(c^*))$. In [17, Prop. 9] a canonical section σ_{c,c^*} of \mathcal{D} is constructed, whose zero set is supported on

$$\mathcal{D} := \{([\mathcal{F}], [\mathcal{G}]) \in M_H^X(c) \times M_H^X(c^*) \mid H^0(X, \mathcal{F} \otimes \mathcal{G}) \neq 0\}.$$

The element σ_{c,c^*} of $H^0(M_H^X(c), \mathcal{D}_{c,c^*}) \otimes H^0(M_H^X(c^*), \mathcal{D}_{c^*,c})$, gives a linear map

$$(2.10) \quad SD_{c,c^*} : H^0(M_H^X(c), \mathcal{D}_{c,c^*})^\vee \rightarrow H^0(M_H^X(c^*), \mathcal{D}_{c^*,c}),$$

called the *strange duality map*. Le Potier's strange duality is then the following.

Conjecture/Question 2.11. Is SD_{c,c^*} an isomorphism?

3. WALLCROSSING FORMULA

3.1. Theta functions and modular forms. For $\tau \in \mathcal{H} = \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$ put $q = e^{\pi i \tau / 4}$ and for $h \in \mathbb{C}$ put $y = e^{h/2}$. Note that the notation is not standard. Recall the 4 theta functions:

$$(3.1) \quad \begin{aligned} \theta_1(h) &:= \sum_{n \in \mathbb{Z}} i^{2n-1} q^{(2n+1)^2} y^{2n+1}, & \theta_2(h) &:= \sum_{n \in \mathbb{Z}} q^{(2n+1)^2} y^{2n+1}, \\ \theta_3(h) &:= \sum_{n \in \mathbb{Z}} q^{(2n)^2} y^{2n}, & \theta_4(h) &:= \sum_{n \in \mathbb{Z}} i^{2n} q^{(2n)^2} y^{2n}. \end{aligned}$$

We usually do not denote the argument τ . If necessary we write $\theta_i(\tau|h)$. The conventions are essentially the same as in [30] and in [2], where the θ_i for $i \leq 3$ are denoted ϑ_i and θ_4 is however denoted ϑ_0 . Denote

$$(3.2) \quad \theta_i := \theta_i(0), \quad \tilde{\theta}_i(h) := \frac{\theta_i(h)}{\theta_i}, \quad i = 2, 3, 4; \quad \tilde{\theta}_1(h) := \frac{\theta_1(h)}{\theta_4}$$

the corresponding Nullwerte and the normalized theta functions. The theta functions satisfy the Jacobi identity

$$(3.3) \quad \theta_3^4 = \theta_2^4 + \theta_4^4,$$

as well as the quadratic relations

$$(3.4) \quad \begin{pmatrix} 0 & \theta_2^2 & -\theta_3^2 & \theta_4^2 \\ -\theta_2^2 & 0 & -\theta_4^2 & \theta_3^2 \\ -\theta_3^2 & -\theta_4^2 & 0 & \theta_2^2 \\ \theta_4^2 & \theta_3^2 & -\theta_2^2 & 0 \end{pmatrix} \begin{pmatrix} \theta_1(h)^2 \\ \theta_2(h)^2 \\ \theta_3(h)^2 \\ \theta_4(h)^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We define a modular function

$$u = -\frac{\theta_2^2}{\theta_3^2} - \frac{\theta_3^2}{\theta_2^2} = \frac{1}{4q^2}(-1 - 20q^4 + 62q^8 - 216q^{12} + 641q^{16} + \dots),$$

and two Jacobi functions (Jacobi forms of weight and index 0) by

$$\begin{aligned} \Lambda &:= \frac{\theta_1(h)}{\theta_4(h)} = -i(y - y^{-1})q - i(y^3 - y + y^{-1} - y^{-3})q^5 + \dots, \\ M &:= 2 \frac{\tilde{\theta}_2(h)\tilde{\theta}_3(h)}{\tilde{\theta}_4(h)^2} = (y + y^{-1}) + 3(y^3 - y - y^{-1} + y^{-3})q^4 + \dots \end{aligned}$$

The quadratic relations above imply

$$\frac{M^2}{4} = \left(\frac{\theta_4^2 \theta_2(h)^2}{\theta_2^2 \theta_4(h)^2} \right) \left(\frac{\theta_4^2 \theta_3(h)^2}{\theta_3^2 \theta_4(h)^2} \right) = \left(1 - \frac{\theta_3^2}{\theta_2^2} \Lambda^2 \right) \left(1 - \frac{\theta_2^2}{\theta_3^2} \Lambda^2 \right) = (1 + u\Lambda^2 + \Lambda^4),$$

and comparing the coefficients of q^0 gives that $M = 2\sqrt{1 + u\Lambda^2 + \Lambda^4}$. We also have the relation

$$(3.5) \quad \frac{\partial \Lambda}{\partial h} = \frac{\theta_2 \theta_3}{4i} M,$$

which follows from [2, §26], and which is equivalent to the formula

$$(3.6) \quad \begin{aligned} h &= \frac{2i}{\theta_2 \theta_3} \int_0^\Lambda \frac{dx}{\sqrt{1 + ux^2 + x^4}} \\ &= i(q^{-1} - 2q^3 + 3q^7 + \dots)\Lambda + i\left(\frac{1}{24}q^{-3} + \frac{3}{4}q - \frac{33}{8}q^5 + \dots\right)\Lambda^3 + \dots \end{aligned}$$

We have the power series developments

$$(3.7) \quad \begin{aligned} \frac{1}{\sqrt{1 + u\Lambda^2 + \Lambda^4}} &= \sum_{\substack{n \geq 0 \\ n \geq k \geq 0}} \binom{-\frac{1}{2}}{n} \binom{n}{k} u^k \Lambda^{4n-2k}, \\ h &= \frac{2i}{\theta_2 \theta_3} \cdot \sum_{\substack{n \geq 0 \\ n \geq k \geq 0}} \binom{-\frac{1}{2}}{n} \binom{n}{k} \frac{u^k \Lambda^{4n-2k+1}}{4n-2k+1}. \end{aligned}$$

Notation 3.8. (1) We denote $\mathcal{R} := \mathbb{Q}[[q^2\Lambda^2, q^4]]$.

(2) Let $\mathbb{Q}[t_1, \dots, t_k]_n$ be the set of polynomials in t_1, \dots, t_k of degree n and $\mathbb{Q}[t_1, \dots, t_k]_{\leq n}$ the set of polynomials of degree at most n .

The formula (3.6) together with the definition of u show that $h \in q^{-1}\Lambda\mathcal{R}$. A function $F(\tau, h)$ can via formula (3.6) also be viewed as a function of τ and Λ . In this case, viewing τ and Λ as the independent variables we define

$$F' := \frac{4}{\pi i} \frac{\partial F}{\partial \tau} = q \frac{\partial F}{\partial q}, \quad F^* := \Lambda \frac{\partial F}{\partial \Lambda}.$$

Thus (3.5) and a simple calculation give that

$$(3.9) \quad h^* = \frac{4i\Lambda}{\theta_2 \theta_3 M}, \quad u' = \frac{2\theta_4^8}{\theta_2^2 \theta_3^2}.$$

Remark 3.10. The natural set of variables for working with elliptic functions is (τ, h) . We will see in a moment that the wallcrossing for the K -theoretic Donaldson invariants is given by a formula in modular forms and elliptic functions, expressed in terms of τ and Λ . In order to prove properties of the wallcrossing formula we usually have to work with the natural variables (τ, h) and then translate the result back into the variables (τ, Λ) . Thus the interplay between the two sets of variables (τ, h) and (τ, Λ) is an important theme in this work.

3.2. Wallcrossing formula. Now we review the wallcrossing formula from [11]. Let $\sigma(X)$ be the signature of X . Fix $c_1 \in H^2(X, \mathbb{Z})$. Let $L \in \text{Pic}(X)$ with $\langle c_1(L), c_1 \rangle$ even. Let $\xi \in H^2(X, \mathbb{Z})$ with $\xi^2 \leq 0$ and $\xi - c_1 \in 2H^2(X, \mathbb{Z})$.

Definition 3.11. Let

$$\Delta_\xi^X(L) := 2i^{\langle \xi, K_X \rangle} \Lambda^2 q^{-\xi^2} y^{\langle \xi(L-K_X) \rangle} \widetilde{\theta}_4(h)^{(L-K_X)^2} \theta_4^{\sigma(X)} u' h^*.$$

By the results of the previous section it can be developed as a power series

$$\Delta_\xi^X(L) = \sum_{d \geq 0} f_d(\tau) \Lambda^d \in \mathbb{Q}((q))[[\Lambda]],$$

whose coefficients $f_d(\tau)$ are Laurent series in q . The *wallcrossing term* is

$$\delta_\xi^X(L) := \sum_{d \geq 0} \delta_{\xi,d}^X(L) \Lambda^d \in \mathbb{C}[[\Lambda]],$$

with

$$\delta_{\xi,d}^X(L) = \text{Coeff}_{q^0}[f_d(\tau)].$$

Setup: For the rest of section 3.2 let H_- , H_+ be ample divisors on X , which do not lie on a wall of type (c_1, d) . Let B_+ be the set of classes ξ of type (c_1, d) with $\langle \xi, H_+ \rangle > 0 > \langle \xi, H_- \rangle$.

The main result of [11] is the following.

Theorem 3.12.

$$\chi(M_{H_+}^X(c_1, d), \mu(L)) - \chi(M_{H_-}^X(c_1, d), \mu(L)) = \sum_{\xi \in B_+} \delta_{\xi,d}^X(L).$$

This is a combination of Prop. 2.11, Cor. 4.2 and Thm. 4.3 in [11] together with the results of Section 4.4 in [11]. In [11, Cor. 4.2] one has to take $v = -v(L)$, thus $\langle \xi, (K_X + c_1(v) + \frac{\text{rk}(v)}{2}(c_1 - K_X)) \rangle = \langle \xi, (K_X - L) \rangle$ and $(K_X + c_1(v) + \frac{\text{rk}(v)}{2}(c_1 - K_X))^2 = (L - K_X)^2$. The results of [11] apply to what are called there *good* walls. However our assumption that $-K_X$ is ample implies that all walls on X are good.

Note that, as X is a simply connected surface with $p_g = 0$, the Euler number $e(X)$ and the signature $\sigma(X)$ are related by $e(X) + \sigma(X) = 4$, thus in [11, Cor. 4.2] one has

$$\exp(e(X)A + \sigma(X)B) = \frac{4\theta_4^{\sigma(X)}}{\theta_2^2 \theta_3^3}.$$

Note that the u of [11] corresponds to $u\Lambda^2$ in the current paper, and the function U_1 of [11] is denoted by M here. Furthermore d in [11] corresponds to $d - 3$ here. Furthermore by definition $\theta_1(h)$ is equal to $-\theta_{11}(\frac{h}{2\pi i}, \tau)$ in [11]. We put $\beta = 1$ in the results of [11]. Thus Thm. 4.3 and the results of Section 4.4. of [11] give that Theorem 3.12 is true if we replace $\Delta_\xi^X(L)$ by

$$i^{\langle \xi K_X \rangle} \Lambda^3 q^{-\xi^2} e^{\langle \xi(L - K_X) \rangle h/2} \left(\frac{\theta_1(h)}{\Lambda \theta_4} \right)^{(L - K_X)^2} \frac{16i\theta_4^{\sigma(X)+8}}{\theta_2^3 \theta_3^3 M}.$$

Note that by definition $e^{h/2} = y$, $\frac{\theta_1(h)}{\Lambda \theta_4} = \tilde{\theta}_4(h)$, and finally we have by (3.9) that $u'h^* = \frac{8i\Lambda\theta_4^8}{\theta_2^3 \theta_3^3 M}$. Thus the result follows.

Remark 3.13. Theorem 3.12 also applies to $\text{Coeff}_{\Lambda^4} [\chi_0^{X,H}(L)]$ and thus to all the generating functions $\chi_{c_1}^{X,H}(L)$.

- (1) Let H_1, H_2 be ample on X , assume they do not lie on a wall of type $(0, 4)$. Then

$$\text{Coeff}_{\Lambda^4} [\chi_0^{X, H_1}(L) - \chi_0^{X, H_2}(L)] = \sum_{\xi} \delta_{\xi, 4}^X(L).$$

where ξ runs through all classes of type $(0, 4)$ with $\langle \xi, H_1 \rangle > 0 > \langle \xi, H_2 \rangle$.

- (2) Let $c_1 \in H^2(X, \mathbb{Z})$. Let H_1, H_2 be ample on X , assume they do not lie on a wall of type (c_1) . Then

$$\chi_{c_1}^{X, H_1}(L) - \chi_{c_1}^{X, H_2}(L) = \sum_{\xi} \delta_{\xi}^X(L),$$

where ξ runs through all classes in $c_1 + 2H^2(X, \mathbb{Z})$ with $\langle \xi, H_1 \rangle > 0 > \langle \xi, H_2 \rangle$.

Proof. (2) follows immediately from Theorem 3.12 and (1).

(1) Let $\epsilon > 0$ sufficiently small such that there is no class ξ of type $(E, 5)$ on \widehat{X} with $\langle \xi, H_1 \rangle < 0 < \langle \xi, H_1 - \epsilon E \rangle$ or with $\langle \xi, H_2 \rangle < 0 < \langle \xi, H_2 - \epsilon E \rangle$. Then by Lemma 2.3 we have for $i = 1, 2$ that $\chi(M_{H_i}^{\widehat{X}}(0, 4), \mu(L)) = \chi(M_{H_i - \epsilon E}^{\widehat{X}}(E, 5), \mu(L))$. Thus by Definition 2.5 we need to show

$$\chi(M_{H_1 - \epsilon E}^{\widehat{X}}(E, 5), \mu(L)) - \chi(M_{H_2 - \epsilon E}^{\widehat{X}}(E, 5), \mu(L)) = \sum_{\xi} \delta_{\xi, 4}^X(L),$$

where ξ runs through the classes of type $(0, 4)$ on X with $\langle \xi, H_1 \rangle > 0 > \langle \xi, H_2 \rangle$. We have

$$(3.14) \quad \chi(M_{H_1 - \epsilon E}^{\widehat{X}}(E, 5), \mu(L)) - \chi(M_{H_2 - \epsilon E}^{\widehat{X}}(E, 5), \mu(L)) = \sum_{\xi'} \delta_{\xi', 5}^{\widehat{X}}(L),$$

where ξ' runs through the classes of type $(E, 5)$ on \widehat{X} with $\langle \xi', H_1 - \epsilon E \rangle > 0 > \langle \xi', H_2 - \epsilon E \rangle$. Note that $H^2(\widehat{X}, \mathbb{Z}) = H^2(X, \mathbb{Z}) \oplus \mathbb{Z}E$. Thus we get that these classes are of the form $\xi' = \xi + (2n - 1)E$ with $n \in \mathbb{Z}$, where ξ is a class of type $(0, 4)$ on X with $\xi'^2 = \xi^2 - (2n - 1)^2 \geq -5$. Note that by definition, if $\xi'^2 < -5$, we get $\delta_{\xi', 5}^{\widehat{X}}(L) = 0$, thus we can replace the sum in (3.14) by the sum over all $\xi' = \xi + (2n - 1)E$ with $n \in \mathbb{Z}$. Finally we note that

$$\sum_{n \in \mathbb{Z}} \delta_{\xi + (2n - 1)E, 5}^{\widehat{X}}(L) = \text{Coeff}_{q^0 \Lambda^5} \left[\sum_{n \in \mathbb{Z}} \Delta_{\xi + (2n - 1)E}^{\widehat{X}}(L) \right],$$

and by Definition 3.11 we have

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \Delta_{\xi + (2n - 1)E}^{\widehat{X}}(L) &= \sum_{n \in \mathbb{Z}} i^{-(2n+1)} q^{(2n+1)^2} y^{2n+1} \widetilde{\theta}_4(h)^{-1} \theta_4 \Delta_{\xi}^X(L) \\ &= \frac{\theta_1(h)}{\theta_4(h)} \Delta_{\xi}^X(L) = \Lambda \Delta_{\xi}^X(L). \end{aligned}$$

Thus $\sum_{n \in \mathbb{Z}} \delta_{\xi + (2n - 1)E, 5}^{\widehat{X}}(L) = \delta_{\xi, 4}^X(L)$. □

Remark 3.15. (1) $\delta_{\xi, d}^X(L) = 0$ unless $d \equiv -\xi^2 \pmod{4}$ (equivalently $d \equiv -c_1^2 \pmod{4}$).

(2) In the definition of $\delta_{\xi}^X(L)$ we can replace $\Delta_{\xi}^X(L)$ by

$$(3.16) \quad \begin{aligned} \overline{\Delta}_{\xi}^X(L) &:= \frac{1}{2}(\Delta_{\xi}^X(L) - \Delta_{-\xi}^X(L)) \\ &= i^{\langle \xi, K_X \rangle} \Lambda^2 q^{-\xi^2} (y^{\langle \xi, L - K_X \rangle} - (-1)^{\xi^2} y^{-\langle \xi, L - K_X \rangle}) \widetilde{\theta}_4(h)^{(L - K_X)^2} \theta_4^{\sigma(X)} u' h^*. \end{aligned}$$

Proof. (1) As $h \in \mathbb{Q}[[q^{-1}\Lambda, q^4]]$, we also have $h^*, y, \tilde{\theta}_4(h) \in \mathbb{Q}[[q^{-1}\Lambda, q^4]]$. Finally $u, u' \in q^{-2}\mathbb{Q}[[q^4]]$. It follows that $\Delta_\xi^X(L) \in q^{-\xi^2}\mathbb{Q}[[q^{-1}\Lambda, q^4]]$. Writing $\Delta_\xi^X(L) = \sum_d f_{d,r}(\tau)\Lambda^d$, we see that $\text{Coeff}_{q^0}[f_{d,r}(\tau)] = 0$ unless $d \equiv -\xi^2 \pmod{4}$.

(2) Note that $\tilde{\theta}_4$ is even in Λ and h^* is odd in Λ , thus

$$\overline{\Delta}_\xi^X(L) = \sum_{d \equiv -\xi^2 \pmod{2}} f_{d,r}(\tau)\Lambda^d,$$

and the claim follows by (1). \square

3.3. Polynomiality and vanishing of the wallcrossing. By definition the wallcrossing terms $\delta_\xi^X(L)$ are power series in Λ . We now show that they are always polynomials. This has been shown already in [11, Rem. 2.9] using a geometric definition of $\delta_\xi^X(L)$. Here we will give a proof which only uses elementary properties of theta functions. The arguments used here will play an important role in the rest of the paper.

The wallcrossing formula is expressed in terms of an expression in Jacobi theta functions $\theta_i(h)$ and modular forms: we develop this expression as a Laurent series in q and Λ and take the coefficient of q^0 . Note however that the natural variables for this expression would be τ (or q) and the elliptic variable h . Thus for understanding the wallcrossing formula it is important to understand the interplay between the two sets of variables (τ, h) and (q, Λ) .

We have seen above that $h \in q^{-1}\Lambda\mathbb{Q}[[q^{-2}\Lambda^2, q^4]]$, and thus $y = e^{h/2} \in \mathbb{Q}[[q^{-1}\Lambda, q^4]]$. We write

$$(3.17) \quad \zeta := y - y^{-1} = 2 \sinh(h/2).$$

We want to see that as a function of q, Λ the function ζ has only a pole of order 1 in q . It will follow that many of the functions we will encounter are almost regular in q in the sense that as Laurent series in Λ, q , they have only finitely many monomials with non-strictly positive powers in q whose coefficients do not vanish.

Lemma 3.18. (1) $\zeta = y - y^{-1} \in q^{-1}\Lambda\mathcal{R}$.

(2) $\zeta^{-1} \in q\Lambda^{-1}\mathcal{R}$.

(3) For all integers n we have

$$\begin{aligned} \sinh((2n+1)h/2) &\in \mathbb{Q}[q^{-1}\Lambda]_{\leq |2n+1|}\mathcal{R}, \\ \cosh(nh) &\in \mathbb{Q}[q^{-2}\Lambda^2]_{\leq |n|}\mathcal{R}, \\ \sinh(nh)h^* &\in \mathbb{Q}[q^{-2}\Lambda^2]_{\leq |n|}\mathcal{R} \\ \cosh((2n+1)h/2)h^* &\in \mathbb{Q}[q^{-1}\Lambda]_{\leq |2n+1|}\mathcal{R}. \end{aligned}$$

(4) $\tilde{\theta}_4(h) \in \mathcal{R}$, and we have $\tilde{\theta}_4(h) = 1 + q^2\Lambda^2 + O(q^4)$.

Proof. (1) By (3.1) we see immediately that

$$\Lambda = \frac{\theta_1(h)}{\theta_4(h)} = \sum_{n \geq 0} q^{4n+1}(g_n(y) - g_n(y^{-1})) = \sum_{n \geq 0} q^{4n+1}f_n(\zeta)$$

for $g_n(y)$, and thus $f_n(\zeta)$ suitable odd polynomials of degree $2n + 1$, in other words $\Lambda \in q\zeta\mathbb{Q}[[\zeta^2q^4, q^4]]$. Explicitly

$$\Lambda = -i(\zeta q + (\zeta^3 + 2\zeta)q^5 + (\zeta^5 + 3\zeta^3 + \zeta)q^9 + (\zeta^7 + 5\zeta^5 + 7\zeta^3 + 2\zeta)q^{13} + \dots).$$

Thus we can form the inverse power series $\zeta \in q^{-1}\Lambda\mathcal{R}$, i.e.

$$\zeta = i((q^{-1} - 2q^3 + 3q^7 + \dots)\Lambda + (q - 5q^5 + \dots)\Lambda^3 + (2q^3 - 17q^7 + \dots)\Lambda^5 + \dots).$$

(2) The coefficient l_1 of Λ^1 of ζ is a Laurent series in q starting with iq^{-1} , and $\frac{\zeta}{f_1} \in 1 + q^2\Lambda^2\mathcal{R}$. Thus we have $\frac{1}{\zeta} = \frac{1}{f_1} \frac{f_1}{\zeta} \in q\Lambda^{-1}\mathcal{R}$.

(3) Let $n \in \mathbb{Z}$. From the definition $\zeta = \sinh(h/2) = y - y^{-1}$, $\sinh((2n+1)h/2) = y^{2n+1} - y^{-(2n+1)}$, $\cosh(nh) = y^n + y^{-n}$, we see immediately that $\sinh((2n+1)h/2)$ is an odd polynomial of degree $|2n+1|$ in ζ and $\cosh(nh)$ is even of degree $|2n|$ in ζ . Thus $\sinh((2n+1)h/2) \in q^{-1}\Lambda\mathbb{Q}[q^{-2}\Lambda^2]_{\leq |n|}\mathcal{R}$, $\cosh(nh) \in \mathbb{Q}[q^{-2}\Lambda^2]_{\leq |n|}\mathcal{R}$. Finally note that $\sinh(nh)h^* = \frac{1}{n}\cosh(nh)$ for $n \neq 0$ and $\sinh(0h/2)h^* = 0$ and $\cosh((2n+1)h/2)h^* = \frac{2}{(2n+1)}\sinh((2n+1)h/2)$. .

(4) From the formula

$$\theta_4(h) = 1 + 2 \sum_{n \in \mathbb{Z}_{>0}} (-1)^n q^{4n^2} \cosh(nh),$$

it therefore follows that $\tilde{\theta}_4(h) \in \mathcal{R}$, and we also easily see $\tilde{\theta}_4(h) = 1 + q^2\Lambda^2 + O(q^4)$. \square

Theorem 3.19. (1) $\delta_{\xi,d}^X(L) = 0$ unless $-\xi^2 \leq d \leq \xi^2 + 2|\langle \xi, L - K_X \rangle| + 4$. In particular $\delta_{\xi}^X(L) \in \mathbb{Q}[\Lambda]$.

(2) $\delta_{\xi}^X(L) = 0$ unless $-\xi^2 \leq |\langle \xi, L - K_X \rangle| + 2$. (Recall that by definition $\xi^2 < 0$).

Proof. Let $N := \langle \xi, L - K_X \rangle$. Note that by the condition that $\langle L, \xi \rangle$ is even, we have $(-1)^{\xi^2} = (-1)^N$. Assume first that N is even. Then by (3.16) we have

$$\overline{\Delta}_{\xi}^X(L) = q^{-\xi^2} i^{\langle \xi, K_X \rangle} \Lambda^2 \sinh(Nh/2) h^* \tilde{\theta}_4^{(L-K_X)^2} \theta_4^{\sigma(X)} u'.$$

If $N = 0$, then $\Delta_{\xi}^X(L) = 0$. Now let $N \neq 0$. Then we note that by Lemma 3.18 we have $\sinh(Nh/2)h^* \in q^{-|N|}\Lambda^{|N|}\mathcal{R}$. Note that $\Lambda^2 u' \in q^{-2}\Lambda^2\mathcal{R}$. Putting this together we get

$$\Delta_{\xi}^X(L) \in q^{-\xi^2} q^{-|N|-2} \Lambda^{|N|+2} \mathcal{R}.$$

In case N is odd, we have

$$\overline{\Delta}_{\xi}^X(L) = q^{-\xi^2} i^{\langle \xi, K_X \rangle} \Lambda^2 \cosh(Nh/2) h^* \tilde{\theta}_4^{(L-K_X)^2} \theta_4^{\sigma(X)} u'.$$

Again using $\cosh(Nh/2)h^* \in q^{-|N|}\Lambda^{|N|}\mathcal{R}$, a similar argument shows that $\Delta_{\xi}^X(L) \in q^{-\xi^2} q^{-|N|-2} \Lambda^{|N|+2} \mathcal{R}$. Thus $\delta_{\xi,d}^X(L) = 0$ unless $-\xi^2 - \min(d, 2|N| + 4 - d) \leq 0$, i.e. unless $-\xi^2 \leq d \leq \xi^2 + 2|N| + 4$. In particular $\delta_{\xi}^X(L) = 0$ unless $-\xi^2 \leq \xi^2 + 2|N| + 4$, i.e. unless $-\xi^2 \leq |N| + 2$. \square

Remark 3.20. Theorem 3.19 implies that for any ξ the generating function $\delta_{\xi}^X(L)$ for the wallcrossing terms $\delta_{\xi,d}^X(L)$ can be obtained by a finite simple computation. We know that $\delta_{\xi,d}^X(L) = 0$ for $d > \xi^2 + 2|\langle \xi, L - K_X \rangle| + 4$. Thus to determine $\delta_{\xi}^X(L)$ we only need to compute

the coefficient of q^0 of the coefficients of Λ^d with $d \leq \xi^2 + 2|\langle \xi, L - K_X \rangle| + 4$ of $\overline{\Delta}_\xi^X(L)$, this is a finite computation with the formal Laurent series involved.

Theorem 3.19 implies that the generating functions $\sum_{d>0} \chi(M_H^X(c_1, d), \mu(L))$ are essentially independent of H : If H_1, H_2 are two ample line bundles on X , then $\chi_{c_1}^{X, H_1}(L) - \chi_{c_1}^{X, H_2}(L)$ is a polynomial in Λ , i.e. $\chi(M_{H_1}^X(c_1, d), \mu(L)) = \chi(M_{H_2}^X(c_1, d), \mu(L))$ for all d which are sufficiently large.

3.4. The case of rational ruled surfaces. We investigate the dependence of the polarization of the K -theoretic Donaldson invariants in the case of $\mathbb{P}^1 \times \mathbb{P}^1$ and $\widehat{\mathbb{P}}^2$. We write $F = H - E$ for the class of a ruling on $\widehat{\mathbb{P}}^2$ and $G = (H + E)/2$. Theorem 3.19 implies that the generating functions $\sum_{d>0} \chi(M_H^X(c_1, d), \mu(L))$ are essentially independent of H : If H_1, H_2 are two ample line bundles on X , then $\chi_{c_1}^{X, H_1}(L) - \chi_{c_1}^{X, H_2}(L)$ is a polynomial in Λ , i.e. $\chi(M_{H_1}^X(c_1, d), \mu(L)) = \chi(M_{H_2}^X(c_1, d), \mu(L))$ for all d which are sufficiently large.

Proposition 3.21. *Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ or $X = \widehat{\mathbb{P}}^2$, Let L be a line bundle on X , let $c_1 \in H^2(X, \mathbb{Z})$, and let H_1, H_2 be ample on X . Then there exists a $d_0 > 0$, such that*

$$\chi_{c_1}^{X, H_1}(L) \equiv \chi_{c_1}^{X, H_2}(L) \pmod{\Lambda^{d_0}}.$$

Proof. We write $L = aF + bG$ with $a, b \in \frac{1}{2}\mathbb{Z}$, and $H_i = n_i F + G$ for $i = 1, 2$, and $n_i \in \mathbb{Q}_{>0}$. Then $L - K_X = (a + 2)F + (b + 2)G$. We can assume $n_1 > n_2$. The classes ξ of type (c_1) with $\langle \xi, H_1 \rangle < 0 < \langle \xi, H_2 \rangle$ are of the form $\xi = \alpha F - \beta G$ with $\alpha, \beta \in \frac{1}{2}\mathbb{Z}_{>0}$ satisfying $\beta n_2 < \alpha < \beta n_1$. Assume $\delta_\xi^X(L) \neq 0$. Then we get by Theorem 3.19

$$-\xi^2 = 2\alpha\beta \leq |(a + 2)\beta - (b + 2)\alpha| + 2 \leq (|a| + 2)\beta + (|b| + 2)\alpha + 2 \leq (|a| + 4)\beta + (|b| + 4)\alpha.$$

Therefore $\alpha \leq (|a| + 4)$ or $\beta \leq (|b| + 4)$. In the first case $0 < \beta \leq \frac{|a|+4}{n_2}$, in the second $0 < \alpha \leq n_1\beta$. In both cases α and β are bounded, thus, as $\alpha, \beta \in \frac{1}{2}\mathbb{Z}$, we see that there are only finitely many ξ of type (c_1) with $\langle \xi, H_1 \rangle < 0 < \langle \xi, H_2 \rangle$ and $\delta_\xi^X(L) \neq 0$. Let d_0 be the maximum over these classes of $\xi^2 + 2|\langle \xi, L - K_X \rangle| + 4$. Then $\chi(M_{H_1}^X(c_1, d), \mu(L)) = \chi(M_{H_2}^X(c_1, d), \mu(L))$ for all $d > d_0$. \square

For the line bundles we consider above we can be more specific.

Proposition 3.22. (1) *On $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $X = \widehat{\mathbb{P}}^2$, let $L = nF + lG$, with $0 \leq l \leq 2$ and $n \geq 0$. We write an ample divisor on $\mathbb{P}^1 \times \mathbb{P}^1$ or $\widehat{\mathbb{P}}^2$ as $aF + bG$. Then $\chi_0^{X, aF+bG}(L)$ and $\chi_F^{X, aF+bG}(L)$ are independent of the ample class $aF + bG$ as long as $\frac{a}{b} > \frac{n+2}{4}$.*
 (2) *If $0 \leq n \leq 3$, $0 \leq e \leq \min(n, 2)$ we have $\chi_0^{\widehat{\mathbb{P}}^2, P}(nH - eE)$ and $\chi_F^{\widehat{\mathbb{P}}^2, P}(nH - eE)$ are independent of the ample class P on $\widehat{\mathbb{P}}^2$.*

Proof. (1) Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ or $X = \widehat{\mathbb{P}}^2$. Let $c_1 = 0$ or $c_1 = F$. Then the classes ξ of type (c_1) on X with $\langle F, \xi \rangle < 0$ can be written as $\xi = aF - bG$ with a, b positive integers and b even. ξ is orthogonal to $P = aF + bG$. If $\delta_\xi^X(L) \neq 0$, then by Theorem 3.19 we get

$$-(aF - bG)^2 \leq |(aF - bG)((n + 2)F + (l + 2)G)| + 2,$$

i.e. $2ab \leq a(l+2) - b(n+2) + 2$ or $2ab \leq -a(l+2) + b(n+2) + 2$. By $b \geq 2$ and $n \geq 0$, we have $-b(n+2) + 2 < 0$; and by $b \geq 2$, $0 \leq l \leq 2$ we have $2ab \geq a(l+2)$. Thus $2ab > a(l+2) - b(n+2) + 2$. On the other hand again by $a \geq 1, b \geq 2$, the inequality $2ab \leq -a(l+2) + b(n+2) + 2$ implies $2a \leq (n+2)$. As $b \geq 2$, this implies that $\frac{a}{b} \leq \frac{n+2}{4}$.

(3) Now let $X = \widehat{\mathbb{P}}^2$, and $\xi = aF - bG$ a class of type (0) or of type (F), with $\langle \xi, F \rangle < 0$. Then $a \geq 1$ and b is even with $b \geq 2$. Let $L = nF + lG$ with $l \leq 2, n \leq 3$. We have seen above that $a \leq \frac{n+2}{2}$. Thus, if $n \leq 1$, we get $a = 1$, but $b \geq 2$ implies that $\xi = (1 - b/2)H - (1 + b/2)E$ is not orthogonal to an ample class. If $2 \leq n \leq 3$, we get $a \leq 2$, and we find that the only $\xi = aF - bG$ orthogonal to an ample class is $2F - 2G = H - 3E$. This is a class of type (F), not of type (0), so the claim follows if $c_1 = 0$. If $c_1 = F$, the possible values for L are $L = 0$, $L = H - E$, $L = 2H - 2E$, $L = 2H$, $L = 3H - E$. In each of these cases one sees directly that

$$-(H - 3E)^2 > |(H - 3E)(L + 3H - E)| + 2,$$

so that $\delta_{H-3E}^X(L) = 0$. □

4. INDEFINITE THETA FUNCTIONS AND VANISHING AND BLOWUP FORMULAS

4.1. Theta functions for indefinite lattices. We begin by reviewing some notations about modular forms.

Definition 4.1. Let $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL(2, \mathbb{Z})$. Let $\Gamma^0(4) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid b \equiv 0 \pmod{4} \right\}$. We recall that $\Gamma^0(4)$ is generated as a group by $\pm T^4, \pm TST$. Recall that the cusps of $\mathcal{H}/\Gamma^0(4)$ are ∞ (of width 4) and 0, 2 (of width 1).

For $\tau \in \mathcal{H}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^0(4)$, we write $A\tau := \frac{a\tau+b}{c\tau+d}$. A holomorphic function $f : \mathcal{H} \rightarrow \mathbb{C}$ is called a weakly holomorphic modular form of weight k on $\Gamma^0(4)$ if $f(A\tau) = (c\tau + d)^k f(\tau)$ for $\tau \in \mathcal{H}$, $A \in \Gamma^0(4)$, and f is meromorphic at the cusps. We denote $M_k^!(\Gamma^0(4))$ the set of all weakly holomorphic modular forms of weight k on $\Gamma^0(4)$.

For us a *lattice* is a free \mathbb{Z} -module Γ together with a quadratic form $Q : \Gamma \rightarrow \frac{1}{2}\mathbb{Z}$, such that the associated bilinear form $x \cdot y := Q(x+y) - Q(x) - Q(y)$ is nondegenerate and \mathbb{Z} -valued. We denote the extension of the quadratic and bilinear form to $\Gamma_{\mathbb{R}} := \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ and $\Gamma_{\mathbb{C}} := \Gamma \otimes_{\mathbb{Z}} \mathbb{C}$ by the same letters. Later the lattice we consider will be $H^2(X, \mathbb{Z})$ with the negative of the intersection form. We will then denote $\langle F, G \rangle$ the intersection form and F^2 the self intersection, and write $F \cdot G$ for the negative of the intersection form.

Now let Γ be a lattice of rank r . Denote by M_{Γ} the set of meromorphic maps $f : \Gamma_{\mathbb{C}} \times \mathcal{H} \rightarrow \mathbb{C}$. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$, we define a map $|_k A : M_{\Gamma} \rightarrow M_{\Gamma}$ by

$$f|_k A(x, \tau) := (c\tau + d)^{-k} \exp \left(-2\pi\sqrt{-1} \frac{cQ(x)}{c\tau + d} \right) f \left(\frac{x}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right).$$

Then $|_k A$ defines an action of $SL(2, \mathbb{Z})$ on M_Γ .

We briefly review the theta functions for indefinite lattices of type $(r-1, 1)$ introduced in [12], when Γ will be $H^2(X, \mathbb{Z})$ for a rational surface X with the negative of the intersection form, and h will be the class of an ample divisor on X .

We denote

$$S_\Gamma := \{f \in \Gamma \mid f \text{ primitive}, Q(f) = 0, f \cdot h < 0\}, \quad C_\Gamma := \{m \in \Gamma_\mathbb{R} \mid Q(m) < 0, m \cdot h < 0\}.$$

For $f \in S_\Gamma$ put $D(f) := \{(\tau, x) \in \mathcal{H} \times \Gamma_\mathbb{C} \mid 0 < \Im(f \cdot x) < \Im(\tau)\}$, and for $h \in C_\Gamma$ put $D(h) = \mathcal{H} \times \Gamma_\mathbb{C}$. For $t \in \mathbb{R}$ denote

$$\mu(t) := \begin{cases} 1 & t \geq 0, \\ 0 & t < 0. \end{cases}$$

Let $c, b \in \Gamma$. Let $f, g \in S_\Gamma \cup C_\Gamma$. Then for $(\tau, x) \in D(f) \cap D(g)$ define

$$\Theta_{\Gamma, c, b}^{f, g}(\tau, x) := \sum_{\xi \in \Gamma + c/2} (\mu(\xi \cdot f) - \mu(\xi \cdot g)) e^{2\pi i \tau Q(\xi)} e^{2\pi i \xi \cdot (x + b/2)}.$$

In [12] it is shown that this sum converges absolutely and locally uniformly on $D(f) \cap D(g)$. Furthermore the following are shown:

Remark 4.2. For $f \in S_\Gamma$, $g \in C_\Gamma$ the function $\Theta_{X, c, b}^{f, g}(\tau, x)$ has a meromorphic continuation to $|\Im(f \cdot x)/\Im(\tau)| < 1$, given by the Fourier expansion.

$$\sum_{\xi \cdot f \neq 0} (\mu(\xi \cdot f) - \mu(\xi \cdot g)) e^{2\pi i \tau Q(\xi)} e^{2\pi i \xi \cdot (x + b/2)} + \frac{1}{1 - e^{2\pi i f \cdot (x + b/2)}} \sum_{\substack{\xi \cdot f = 0 \\ f \cdot g \leq \xi \cdot g < 0}} e^{2\pi i \tau Q(\xi)} e^{2\pi i \xi \cdot (x + b/2)},$$

with the sums running through $\xi \in \Gamma + c/2$.

Theorem 4.3. (1) For $f, g \in S_\Gamma$ the function $\Theta_{X, c, b}^{f, g}(\tau, x)$ has a meromorphic continuation to $\mathcal{H} \times \Gamma_\mathbb{C}$.

(2) For $|\Im(f \cdot x)/\Im(\tau)| < 1$ and $|\Im(g \cdot x)/\Im(\tau)| < 1$ it has a Fourier development

$$\begin{aligned} \Theta_{X, c, b}^{f, g}(x, \tau) &:= \frac{1}{1 - e^{2\pi i f \cdot (x + b/2)}} \sum_{\substack{\xi \cdot f = 0 \\ f \cdot g \leq \xi \cdot g < 0}} e^{2\pi i \tau Q(\xi)} e^{2\pi i \xi \cdot (x + b/2)} \\ &- \frac{1}{1 - e^{2\pi i g \cdot (x + b/2)}} \sum_{\substack{\xi \cdot g = 0 \\ f \cdot g \leq \xi \cdot f < 0}} e^{2\pi i \tau Q(\xi)} e^{2\pi i \xi \cdot (x + b/2)} + \sum_{\xi \cdot f > 0 > \xi \cdot g} e^{2\pi i \tau Q(\xi)} (e^{2\pi i \xi \cdot (x + b/2)} - e^{-2\pi i \xi \cdot (x + b/2)}), \end{aligned}$$

where the sums are always over $v \in \Gamma + c/2$.

(3) (parity) $\Theta_{X, c, b}^{f, g}(-x, \tau) = -(-1)^{c \cdot b} \Theta_{X, c, b}^{f, g}(x, \tau)$,

(4) (*modular properties*)

$$\begin{aligned} (\Theta_{X,c,b}^{f,g} \theta_3^{\sigma(\Gamma)})|_1 S &= (-1)^{-b \cdot c/2} \Theta_{X,b,c}^{f,g} \theta_3^{\sigma(\Gamma)}, \\ (\Theta_{X,c,b}^{f,g} \theta_3^{\sigma(\Gamma)})|_1 T &= (-1)^{3Q(c)/2 - cw/2} \Theta_{X,c,b-c+w}^{f,g} \theta_4^{\sigma(\Gamma)}, \\ (\Theta_{X,c,b}^{f,g} \theta_3^{\sigma(\Gamma)})|_1 T^2 &= (-1)^{-Q(c)} \Theta_{X,c,b}^{f,g} \theta_3^{\sigma(\Gamma)}, \\ (\Theta_{X,c,b}^{f,g} \theta_3^{\sigma(\Gamma)})|_1 T^{-1} S &= (-1)^{-Q(c)/2 - c \cdot b/2} \Theta_{X,w-c+b,c}^{f,g} \theta_2^{\sigma(\Gamma)}, \end{aligned}$$

where w is a characteristic element of Γ .

Remark 4.4. For $f, g, h \in C_\Gamma \cap S_\Gamma$ we have the cocycle condition: $\Theta_{\Gamma,c,b}^{f,g}(\tau, x) + \Theta_{\Gamma,c,b}^{g,h}(\tau, x) = \Theta_{\Gamma,c,b}^{f,h}(\tau, x)$, which holds wherever all three terms are defined.

In the following let X be a rational algebraic surface. We assume for simplicity that $-K_X$ is ample on X . We can express the difference of the K -theoretic Donaldson invariants for two different polarizations in terms of these indefinite theta functions. Here we take Γ to be $H^2(X, \mathbb{Z})$ with the negative of the intersection form. In the formulas above we will take the characteristic element to be K_X .

Definition 4.5. Let $F, G \in S_\Gamma \cup C_\Gamma$, let $c_1 \in H^2(X, \mathbb{Z})$. We put

$$\Psi_{X,c_1}^{F,G}(L; \Lambda, \tau) := \Theta_{X,c_1,K_X}^{F,G} \left(\frac{(L - K_X)h}{2\pi i}, \tau \right) \Lambda^2 \tilde{\theta}_4(h)^{(L-K_X)^2} \theta_4^{\sigma(X)} u' h^*.$$

Corollary 4.6. Let H_1, H_2 be ample on X , and assume that they do not lie on a wall of type (c_1) . Then

$$\chi_{c_1}^{X,H_2}(L) - \chi_{c_1}^{X,H_1}(L) = \text{Coeff}_{q^0} [\Psi_{X,c_1}^{H_1,H_2}(L; \Lambda, \tau)].$$

Proof. We write $J := \Lambda^2 \tilde{\theta}_4(h)^{(L-K_X)^2} \theta_4^{\sigma(X)} u' h^*$. Now assume that ξ is congruent to c_1 modulo $2H^2(X, \mathbb{Z})$. Then it follows that $\delta_{\xi,d}^X = 0$ unless $d + \xi^2 \geq 0$ and d is congruent to $-c_1^2$ modulo 4, i.e. unless ξ is a class of type (c_1, d) . Therefore Theorem 3.12, Remark 3.13 and Remark 3.15 imply that

$$\begin{aligned} \chi_{c_1}^{X,H_2}(L) - \chi_{c_1}^{X,H_1}(L) &= \sum_{H_2 \xi > 0 > H_1 \xi} \text{Coeff}_{q^0} [\overline{\Delta}_\xi^X(L)] \\ &= \text{Coeff}_{q^0} \left[\sum_{H_2 \xi > 0 > H_1 \xi} i^{\langle \xi, K_X \rangle} q^{-\xi^2} (e^{\langle \frac{\xi}{2}, L-K_X \rangle h} - (-1)^{\xi K_X} e^{\langle -\frac{\xi}{2}, L-K_X \rangle h}) J \right] \\ &= \text{Coeff}_{q^0} \left[\Theta_{X,c_1,K_X}^{H_1,H_2} \left(\frac{(L - K_X)h}{2\pi \sqrt{-1}}, \tau \right) J \right] \end{aligned}$$

The sums are over $\xi \in 2H^2(X, \mathbb{Z}) + c_1$. □

We use Corollary 4.6 to extend the generating function $\chi_{c_1}^{X,H}(L)$ formally to $H \in S_L \cup C_L$.

Definition 4.7. Let M be ample on X and not on a wall of type (c_1) . Let $H \in S_X \cup C_X$. We put

$$\chi_{c_1}^{X,H}(L) := \chi_{c_1}^{X,M}(L) + \text{Coeff}_{q^0} [\Psi_{X,c_1}^{M,H}(L; \Lambda, \tau)].$$

By the cocycle condition, the definition of $\chi_{c_1}^{X,M}(L)$ is independent of the choice of H . Furthermore by Corollary 4.6 this coincides with the previous definition in case M is also ample and does not lie on a wall of type (c_1) .

Remark 4.8. Let H be ample on X , possibly lying on a wall of type (c_1, d) , let H_0 be ample on X in an adjacent chamber of type (c_1, d) , i.e. H_0 does not lie on a wall of type (c_1, d) and there are no classes of type (c_1, d) with $\langle \xi, H_0 \rangle < 0 < \langle \xi, H \rangle$. Then the definition of $\Theta_{\Gamma, c, b}^{f, g}$ implies that

$$\text{Coeff}_{\Lambda^d} [\chi_{c_1}^{X, H}(L)] = \text{Coeff}_{\Lambda^d} [\chi_{c_1}^{X, H_0}(L)] + \frac{1}{2} \sum_{\xi} \delta_{\xi, d}^X(L),$$

with ξ running over all classes of type (c_1, d) with $\langle \xi, H_0 \rangle < 0 = \langle \xi, H \rangle$.

4.2. Extension of blowup formulas. Now we will extend Lemma 2.3 to $\chi_{c_1}^{X, H}(L)$ for $H \in C_X \cup S_X$. We will continue to denote a class in $H^2(X, \mathbb{Z})$ and its pullback to the blowup \hat{X} of X in a point by the same letter.

Proposition 4.9. *Let X be a rational surface. Let $H \in C_X \cup S_X$. Let $c_1 \in H^2(X, \mathbb{Z})$. Let \hat{X} be the blowup of X in a general point, and E the exceptional divisor. Assume $-K_{\hat{X}}$ is ample. Let $L \in \text{Pic}(X)$ with $\langle L, c_1 \rangle$ even.*

- (1) $\chi_{c_1}^{\hat{X}, H}(L) = \chi_{c_1}^{X, H}(L),$
- (2) $\chi_{c_1 + E}^{\hat{X}, H}(L) = \Lambda \chi_{c_1}^{X, H}(L).$

Proof. We can choose $H_0 \in C_X$, which does not lie on any wall of type (c_1) on X . In case $X = \mathbb{P}^2$ we take H_0 the hyperplane class. If $b_2(X) > 1$ we can take H_0 general in C_X . For each $d > 0$ and all $c_1 \in H^2(X, \mathbb{Z})$ we choose $\epsilon_d > 0$, such that there is no class ξ of type (c_1, d) or of type $(c_1 + E, d + 1)$ on \hat{X} with $\langle \xi, H_0 \rangle < 0 < \langle \xi, H_0 - \epsilon_d E \rangle$. For simplicity we will write $M_{H_0+}^{\hat{X}}(c_1, d) := M_{H_0 - \epsilon_d}^{\hat{X}}(c_1, d)$, $M_{H_0+}^{\hat{X}}(c_1 + E, d + 1) := M_{H_0 - \epsilon_d}^{\hat{X}}(c_1 + E, d + 1)$. Then by Lemma 2.3 we have for $c_1 \notin 2H^2(X, \mathbb{Z})$ or $d > 4$ that

$$(4.10) \quad \begin{aligned} \chi(M_{H_0+}^{\hat{X}}(c_1, d), \mu(L)) &= \chi(M_{H_0}^X(c_1, d), \mu(L)), \\ \chi(M_{H_0+}^{\hat{X}}(c_1 + E, d + 1), \mu(L)) &= \chi(M_{H_0}^X(c_1, d), \mu(L)). \end{aligned}$$

In case $c_1 \in 2H^2(X, \mathbb{Z})$, we see that H_0 does not lie on a wall of type (c_1, d) or $(c_1 + E, d)$, on \hat{X} , and we get

$$\chi(M_{H_0+}^{\hat{X}}(c_1, d), \mu(L)) = \text{Coeff}_{\Lambda^d} [\chi_{c_1}^{\hat{X}, H_0}(L)], \quad \chi(M_{H_0+}^{\hat{X}}(c_1, d), \mu(L)) = \text{Coeff}_{\Lambda^{d+1}} [\chi_{c_1 + E}^{\hat{X}, H_0}(L)].$$

Thus (1) and (2) follow for H_0 in case $c_1 \notin 2H^2(X, \mathbb{Z})$.

Now we deal with the case $c_1 = 0$. By (2.7) and (4.10) and Remark 4.8 we get

$$(4.11) \quad \begin{aligned} \chi_0^{X, H_0}(L) &= \sum_{d > 0} \chi(M_{H_0+}^{\hat{X}}(E, d + 1), \mu(L)) \Lambda^d + (\langle L, K_X \rangle - \frac{K_X^2 + L^2}{2} - 1) \Lambda^4 \\ &= \frac{1}{\Lambda} \left(\chi_E^{\hat{X}, H_0}(L) + \frac{1}{2} \sum_{\xi} \delta_{\xi}^{\hat{X}}(L) \right) + (\langle L, K_X \rangle - \frac{K_X^2 + L^2}{2} - 1) \Lambda^4, \end{aligned}$$

where ξ runs through all classes ξ of type $(E, d+1)$ on \widehat{X} with $\langle \xi, H_0 \rangle = 0$ and $\langle \xi, E \rangle < 0$; in other words $\xi = (2n-1)E$, with $n > 0$. By Theorem 3.19 we get that $\delta_{nE}^{\widehat{X}}(L) = 0$ unless $n^2 \leq n+2$. Thus in (4.11) we only have to consider $\xi = E$, and Theorem 3.19 gives that $\delta_E^{\widehat{X}, d}(L) = 0$ for $d > 5$. Explicit computations with the Fourier developments of the functions occurring in (3.16) give

$$\delta_E^{\widehat{X}}(L) = (-\langle 2K_{\widehat{X}}, L \rangle + (K_{\widehat{X}+L^2}^2 + 3))\Lambda^5 = (-\langle 2K_X, L \rangle + (K_X^2 + L^2 + 2))\Lambda^5.$$

This shows (2) for H_0 and $c_1 = 0$.

Let \widetilde{X} be the blow up of \widehat{X} in a general point. Then we get by (1) in case $c_1 \notin 2H^2(X, \mathbb{Z})$ and by (2) that

$$\chi_0^{X, H_0}(L) = \frac{1}{\Lambda} \chi_E^{\widehat{X}, H_0}(L) = \frac{1}{\Lambda} \chi_E^{\widetilde{X}, H_0}(L) = \chi_0^{\widetilde{X}, H_0}(L).$$

Now for general H in $C_X \cup S_X$ Definition 4.7 gives

$$\chi_{c_1}^{X, H}(L) = \chi_{c_1}^{X, H_0}(L) + \text{Coeff}_{q^0} [\Psi_{X, c_1}^{H_0, H}(L, \Lambda, \tau)], \quad \chi_{c_1}^{\widehat{X}, H}(L) = \chi_{c_1}^{\widehat{X}, H_0}(L) + \text{Coeff}_{q^0} [\Psi_{\widehat{X}, c_1}^{H_0, H}(L, \Lambda, \tau)].$$

By $H^2(\widehat{X}, \mathbb{Z}) = H^2(X, \mathbb{Z}) + \mathbb{Z}E$, we find that

$$\Theta_{\widehat{X}, c_1, K_{\widehat{X}}}^{H_0, H} \left(\frac{1}{2\pi i} (L - K_{\widehat{X}})h, \tau \right) = \Theta_{X, c_1, K_X}^{H_0, H} \left(\frac{1}{2\pi i} (L - K_X)h, \tau \right) \theta_4(h).$$

As $(L - K_{\widehat{X}})^2 = (L - K_X)^2 - 1$, and $\sigma(\widehat{X}) = \sigma(X) - 1$, we get by definition

$$\Psi_{\widehat{X}, c_1}^{H_0, H}(L, \Lambda, \tau) = \Psi_{X, c_1}^{H_0, H}(L, \Lambda, \tau) \frac{\theta_4(h)}{\theta_4(h)\theta_4} = \Psi_{X, c_1}^{H_0, H}(L, \Lambda, \tau).$$

Similarly we get the following

$$\chi_{c_1+E}^{\widehat{X}, H}(L) = \chi_{c_1+E}^{\widehat{X}, H_0}(L) + \text{Coeff}_{q^0} [\Psi_{\widehat{X}, c_1+E}^{H_0, H}(L, \Lambda, \tau)].$$

By $H^2(\widehat{X}, \mathbb{Z}) = H^2(X, \mathbb{Z}) + \mathbb{Z}E$, we find that

$$\Theta_{\widehat{X}, c_1+E, K_{\widehat{X}}}^{H_0, H} \left(\frac{1}{2\pi i} (L - K_{\widehat{X}})h, \tau \right) = \Theta_{X, c_1, K_X}^{H_0, H} \left(\frac{1}{2\pi i} (L - K_X)h, \tau \right) \theta_1(h).$$

Thus we get by definition

$$\Psi_{\widehat{X}, c_1+E}^{H_0, H}(L, \Lambda, \tau) = \Psi_{X, c_1+E}^{H_0, H}(L, \Lambda, \tau) \frac{\theta_1(h)}{\theta_4(h)\theta_4} = \Psi_{X, c_1}^{H_0, H}(L, \Lambda, \tau) \frac{\theta_1(h)}{\theta_4(h)},$$

and use $\Lambda = \frac{\theta_1(h)}{\theta_4(h)}$. This shows the result. \square

4.3. Modularity properties. We want to show that under suitable assumptions the difference of the K -theoretic Donaldson invariants between two points $F, G \in S_X$ vanishes. For this we first show that $\Psi_{X,c_1}^{F,G}(L; \Lambda, \tau)$ has a power series development in Λ , whose coefficients are modular forms of weight 2 on $\Gamma^0(4)$. The result is then proven by replacing the q -development at the cusp ∞ by that at the other two cusps of $\mathcal{H}/\Gamma^0(4)$.

Convention: In this whole section A will always stand for a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$.

Lemma 4.12. *Let $F, G \in S_X$. Then*

$$\Psi_{X,c_1}^{F,G}(L; \Lambda, \tau) \in M_2^!(\Gamma^0(4))[[\Lambda]].$$

In order to prove Lemma 4.12, we first study the transformation behaviour of $\Theta_{X,c_1,K_X}^{F,G}$ under $\Gamma^0(4)$.

Lemma 4.13. (1) $(\Theta_{X,c_1,K_X}^{F,G} \theta_4^{\sigma(X)})|_1 A = (-1)^{(c_1^2)/4 - \langle c_1, K_X \rangle / 2} \Theta_{X,c_1,K_X}^{F,G} \theta_4^{\sigma(X)}$ for $A \in \Gamma^0(4)$.
 (2) $(\Theta_{X,c_1,K_X}^{F,G} \theta_4^{\sigma(X)})|_1 S = i^{\langle c_1, K_X \rangle} \Theta_{X,K_X,c_1}^{F,G} \theta_2^{\sigma(X)}$.

Proof. By using the identities (4) systematically we get

$$\begin{aligned} (\Theta_{X,c_1,K_X}^{F,G} \theta_4^{\sigma(X)})|_1 T^2 &= (-1)^{(c_1^2)/4 - \langle c_1, K_X \rangle / 2} (\Theta_{X,c_1,c_1}^{F,G} \theta_3^{\sigma(X)})|_1 T \\ &= (-1)^{-(c_1^2)/2} \Theta_{X,c_1,K_X}^{F,G} \theta_4^{\sigma(X)}, \end{aligned}$$

In particular $(\Theta_{X,c_1,K_X}^{F,G} \theta_4^{\sigma(X)})|_1 T^4 = (-1)^{c_1^2} \Theta_{X,c_1,K_X}^{F,G} \theta_4^{\sigma(X)}$. Similarly we get

$$\begin{aligned} (\Theta_{X,c_1,K_X}^{F,G} \theta_4^{\sigma(X)})|_1 TST &= (-1)^{c_1^2/4 - \langle c_1, K_X \rangle / 2} (\Theta_{X,c_1,c_1}^{F,G} \theta_3^{\sigma(X)})|_1 ST \\ &= (-1)^{3c_1^2/4 - \langle c_1, K_X \rangle / 2} (\Theta_{X,c_1,c_1}^{F,G} \theta_3^{\sigma(X)})|_1 T \\ &= \Theta_{X,c_1,K_X}^{F,G} \theta_4^{\sigma(X)}. \end{aligned}$$

The last two formulas imply (1). Finally

$$\begin{aligned} (\Theta_{X,c_1,K_X}^{F,G} \theta_4^{\sigma(X)})|_1 S &= (-1)^{c_1^2/4 - \langle c_1, K_X \rangle / 2} (\Theta_{X,c_1,c_1}^{F,G} \theta_3^{\sigma(X)})|_1 T^{-1} S \\ &= (-1)^{c_1^2 - \langle c_1, K_X \rangle / 2} \Theta_{X,K_X,c_1}^{F,G} \theta_2^{\sigma(X)} = (-1)^{\langle c_1, K_X \rangle / 2} \Theta_{X,K_X,c_1}^{F,G} \theta_2^{\sigma(X)}, \end{aligned}$$

where the last equality is because K_X is characteristic, i.e. $(-1)^{c_1^2} = (-1)^{\langle c_1, K_X \rangle}$. \square

Proof of Lemma 4.12. By [2, §22] we have

$$\begin{aligned} (4.14) \quad \tilde{\theta}_4(z)|_0 S &= \frac{\theta_2(z)}{\theta_2}, \quad \frac{\theta_3(z)}{\theta_3}|_0 S = \frac{\theta_3(z)}{\theta_3}, \quad \tilde{\theta}_4(z)|_0 T = \frac{\theta_3(z)}{\theta_3}, \\ \tilde{\theta}_4(z)|_0 T^2 &= \tilde{\theta}_4(z), \quad \tilde{\theta}_4(z)|_0 TST = \frac{\theta_3(z)}{\theta_3}|_0 ST = \frac{\theta_3(z)}{\theta_3}|_0 T = \tilde{\theta}_4(z). \end{aligned}$$

In particular $\tilde{\theta}_4(h)|_0 A = \tilde{\theta}_4(h)$ for $A \in \Gamma^0(4)$. Writing

$$F(z, \tau) := \Theta_{X, c_1, K_X}^{G, F} \left(\frac{(L - K_X)z}{2\pi i}, \tau \right) \theta_4^{\sigma(X)} \tilde{\theta}_4(z)^{(L - K_X)^2},$$

we get by Lemma 4.13 that

$$(4.15) \quad F\left(\frac{z}{c\tau + d}, A\tau\right) = (-1)^{\langle c_1^2 \rangle b/4} (c\tau + d) F(z, \tau), \quad A \in \Gamma^0(4).$$

Again by [2, §22] we see that $u = -\frac{\theta_2^2}{\theta_3^2} - \frac{\theta_3^2}{\theta_2^2}$ is a modular function on $\Gamma^0(4)$, and by definition it is clear that it is holomorphic on \mathcal{H} . We also see from [2, §22] that $\theta_2\theta_3$ transforms under $A \in \Gamma^0(4)$ according to $\theta_2(A\tau)\theta_3(A\tau) = (-1)^{b/4}(c\tau + d)\theta_2(\tau)\theta_3(\tau)$. Thus by (3.7) we see that h transforms under $A \in \Gamma^0(4)$ by

$$(4.16) \quad h(\Lambda, A\tau) = (-1)^{b/4} \frac{h(\Lambda, \tau)}{c\tau + d}.$$

Thus by (4.15) we get for $A \in \Gamma^0(4)$ that

$$(4.17) \quad \begin{aligned} F(h(\Lambda, A\tau), A\tau) &= F\left((-1)^{b/4} \frac{h(\Lambda, \tau)}{c\tau + d}, A\tau\right) = (-1)^{c_1^2 b/4} (c\tau + d) F((-1)^{b/4} h(\Lambda, \tau), \tau) \\ &= (-1)^{b/4} (c\tau + d) F(h(\Lambda, \tau), \tau). \end{aligned}$$

In the last line we use $F(-z, \tau) = -(-1)^{\langle c_1^2 \rangle} F(z, \tau)$, which follows from Theorem 4.3(3) and the fact that $\tilde{\theta}_4(z)$ is even in z . We use (3.9) to write

$$G(\Lambda, \tau) := \Lambda^2 u' h^* = \frac{4i\Lambda^3 \theta_4^8}{\theta_2^3 \theta_3^3} \frac{1}{\sqrt{1 + u\Lambda^2 + \Lambda^4}}.$$

Using (3.7) and the fact that u is a modular function on $\Gamma^0(4)$, we have $G(\Lambda, A\tau) = (-1)^{b/4}(c\tau + d)G(\Lambda, \tau)$ for $A \in \Gamma^0(4)$. Putting all this together, we obtain for $A \in \Gamma^0(4)$ that

$$(4.18) \quad \Psi_{X, c_1}^{F, G}(L; \Lambda, A\tau) = F(h(\Lambda, A\tau), A\tau) G(\Lambda, A\tau) = (c\tau + d)^2 \Psi_{X, c_1}^{F, G}(L; \Lambda, \tau).$$

The Fourier development (3.1) of $\Theta_{X, c_1, K_X}^{G, F}$ and the standard Fourier development of $\theta_4(z)$ imply that we can write F as a formal power series $F = \sum_{n \geq 0} f_n(\tau) z^n$, where each f_n is meromorphic at the cusps and holomorphic on \mathcal{H} . It is easy to see that u is holomorphic on \mathcal{H} and meromorphic at the cusps. We use that $\theta_2, \theta_3, \theta_4$ are holomorphic on \mathcal{H} and at the cusps and without zero on \mathcal{H} . By (3.7) we can write $h(\Lambda, \tau) = \sum_{n \geq 1} h_n(\tau) \Lambda^n$, $G(\Lambda, \tau) = \sum_{n \geq 0} w_n(\tau) \Lambda^n$, where each h_n, w_n is holomorphic on \mathcal{H} and meromorphic at the cusps. Thus we obtain that

$$\Psi_{X, c_1}^{F, G}(L; \Lambda, \tau) = F(h(\Lambda, \tau), \tau) G(\Lambda, \tau) = \sum_{n \geq 0} p_n(\tau) \Lambda^n,$$

where each p_n is holomorphic on \mathcal{H} and meromorphic at the cusps. Thus by (4.18) each p_n is a weakly holomorphic modular form of weight 2 on $\Gamma^0(4)$. \square

4.4. Vanishing of the difference between boundary points. Let again X be a projective surface with $-K_X$ ample. Let $F, G \in S_X$. In this section we want to show that for any c_1, d and any line bundle L on X with $\langle c_1, L \rangle$ even, we have $\chi(M_F^X(c_1, d), \mu(L)) = \chi(M_G^X(c_1, d), \mu(L))$.

Theorem 4.19. *Let $-K_X$ be ample. Let $F, G \in S_X$. Then*

$$\chi_{c_1}^{X,G}(L; \Lambda) - \chi_{c_1}^{X,F}(L; \Lambda) = 0.$$

Proof. By Corollary 4.6 we have to show that $\text{Coeff}_{q^0} [\Psi_{X,c_1}^{F,G}(L; \Lambda, \tau)] = 0$.

By Lemma 4.12 we have $\Psi_{X,c_1}^{F,G}(L; \Lambda, \tau) \in M_2^1(\Gamma^0(4))[[\Lambda]]$, i.e. $[\Psi_{X,c_1}^{F,G}(L; \Lambda, \tau) 2\pi i d\tau]_{\Lambda^d}$ is for all d a holomorphic differential form on $\mathcal{H}/\Gamma^0(4)$, with a meromorphic extension to the cusps $\infty, 0$ and 2 . Note that $2\pi i q d\tau = 8dq$, thus, taking into account that the width of the cusp ∞ is 4, we get by the residue theorem that

$$\begin{aligned} \text{Coeff}_{q^0} [\Psi_{X,c_1}^{F,G}(L; \Lambda, \tau)] &= \text{res}_{\tau=\infty} \left[\Psi_{X,c_1}^{F,G}(L; \Lambda, \tau) \frac{\pi i}{2} d\tau \right] \\ &= -\text{res}_{\tau=0} \left[\Psi_{X,c_1}^{F,G}(L; \Lambda, \tau) \frac{\pi i}{2} d\tau \right] - \text{res}_{\tau=2} \left[\Psi_{X,c_1}^{F,G}(L; \Lambda, \tau) \frac{\pi i}{2} d\tau \right] \\ &= -\frac{1}{4} \text{Coeff}_{q^0} \left[\tau^{-2} \Psi_{X,c_1}^{F,G}(L; \Lambda, S\tau) \right] - \frac{1}{4} \text{Coeff}_{q^0} \left[(\tau - 2)^{-2} \Psi_{X,c_1}^{F,G}(L; \Lambda, S(\tau - 2)) \right]. \end{aligned}$$

By [2, §22] we have

$$(4.20) \quad \theta_2(-1/\tau) = \sqrt{-i\tau} \theta_4(\tau), \quad \theta_3(-1/\tau) = \sqrt{-i\tau} \theta_3(\tau), \quad \theta_3(-1/\tau) = \sqrt{-i\tau} \theta_2(\tau),$$

Again we write

$$\begin{aligned} F(z, \tau) &:= \Theta_{X,c_1,K_X}^{G,F} \left(\frac{(L - K_X)z}{2\pi i}, \tau \right) \theta_4^{\sigma(X)} \tilde{\theta}_4(z)^{(L-K_X)^2}, \\ G(\Lambda, \tau) &:= \Lambda^2 u' h^* = \frac{4i\Lambda^3 \theta_4^8}{\theta_2^3 \theta_3^3} \frac{1}{1 + u\Lambda^2 + \Lambda^4}. \end{aligned}$$

We put $\tilde{u} := u|_0 S$. Then by (4.14) we see that $\tilde{u} = -\frac{\theta_3^4 + \theta_4^4}{\theta_2^3 \theta_3^3} = -2 + O(q^4)$. Let $\tilde{h}(\Lambda, \tau) := \tau h(\Lambda, S\tau)$. By (4.20) we get $(\theta_2 \theta_3)|_1 S = -i\theta_3 \theta_4$. Thus by (3.7) we see that

$$\tilde{h} = -\frac{2}{\theta_2 \theta_3} \sum_{\substack{n \geq 0 \\ n \geq k \geq 0}} \binom{-\frac{1}{2}}{n} \binom{n}{k} \frac{\tilde{u}^k \Lambda^{4n-2k+1}}{4n-2k+1}.$$

In particular both \tilde{u} and \tilde{h} are regular at $q = 0$. We get by Lemma 4.13, (4.20) and (4.14) that

$$\begin{aligned} F\left(\frac{z}{\tau}, S\tau\right) &= (-1)^{\langle c_1, K_X \rangle / 2} \Theta_{X,K_X,c_1}^{G,F} \left(\frac{(L - K_X)z}{2\pi i}, \tau \right) (-1)^{-\sigma(X)/4} \theta_2^{\sigma(X)} \left(\frac{\theta_2(z)}{\theta_2} \right)^{(L-K_X)^2} \\ G(\Lambda, \tau)|_1 S &= \frac{4\Lambda^3 \theta_2^8}{\theta_3^3 \theta_4^3} \frac{1}{1 + \tilde{u}\Lambda^2 + \Lambda^4}. \end{aligned}$$

and

$$F(h(\Lambda, S\tau), S\tau) = (-1)^{\langle c_1, K_X \rangle / 2} \Theta_{X,K_X,c_1}^{G,F} \left(\frac{(L - K_X)\tilde{h}}{2\pi i}, \tau \right) (-1)^{-\sigma(X)/4} \theta_2^{\sigma(X)} \left(\frac{\theta_2(z)}{\theta_2} \right)^{(L-K_X)^2}.$$

Putting this together we see that

$$-\frac{1}{4}\tau^{-2}\Psi_{X,c_1}^{F,G}(L; \Lambda, S\tau) = F(h(\Lambda, S\tau), S\tau) \cdot G(\Lambda, \tau)|_1 S$$

can be written as $\theta_2^{8+\sigma(X)}H(\Lambda, \tau)$, where $H(\Lambda, \tau)$ is regular at $q = 0$. Recall that $-K_X$ is ample, thus $K_X^2 > 0$. As $K_X^2 - \sigma(X) = 8$, this implies $\sigma(X) > -8$. As θ_2 has a zero of order 1 in q , we find that

$$\text{Coeff}_{q=0} \left[-\frac{1}{4}\tau^{-2}\Psi_{X,c_1}^{F,G}(L; \Lambda, S\tau) \right] = 0.$$

Now we compute $\text{Coeff}_{q^0} [(\tau - 2)^{-2}\Psi_{X,c_1}^{F,G}(L; \Lambda, ST^{-2}\tau)]$. It is easy to see that $u(T^{-2}\tau) = -u(\tau)$, thus $u(T^{-2}\tau)\Lambda^2 = (i\Lambda)^2u$ and $\frac{1}{\theta_2\theta_3}|T^{-2} = \frac{i}{\theta_2\theta_3}$. Thus we get by (3.7) that $h(\Lambda, T^{-2}\tau) = h(i\Lambda, \tau)$ and $G(\Lambda, T^{-2}\tau) := i^3G(i\Lambda, \tau)$. We also see $\tilde{\theta}_4(z)|_{T^{-2}} = \tilde{\theta}_4(z)$ and

$$(\Theta_{X,c_1,K_X}^{G,F}((L - K)z, \tau)\theta_4^{\sigma(X)})|_{\tau \rightarrow T^{-2}\tau} = (-1)^{c_1^2/2}\Theta_{X,c_1,K_X}^{G,F}((L - K)z, \tau)\theta_4^{\sigma(X)}$$

Combining these facts, we get $\Psi_{X,c_1}^{F,G}(L; \Lambda, T^{-2}\tau) = i^{c_1^2+3}\Psi_{X,c_1}^{F,G}(L, i\Lambda, \tau)$. Therefore we get

$$(\tau - 2)^{-2}\Psi_{X,c_1}^{F,G}(L; \Lambda, ST^{-2}\tau) = i^{c_1^2+3}\tau^{-2}\Psi_{X,c_1}^{F,G}(L, i\Lambda, S\tau),$$

and thus $\text{Coeff}_{q^0} [\Psi_{X,c_1}^{F,G}(L; \Lambda, ST^{-2}\tau)] = 0$. \square

4.5. Vanishing at boundary of the positive cone. The following standard fact allows us to compute the K -theoretic Donaldson for rational surfaces.

Remark 4.21. Let X be a simply connected algebraic surface, and let $\pi : X \rightarrow \mathbb{P}^1$ be a morphism whose general fibre is isomorphic to \mathbb{P}^1 . Let M be ample on X . Let $F \in H^2(X, \mathbb{Z})$ be the class of a fibre, and assume that $\langle c_1, F \rangle$ is odd. Then for all $d > 0$ there exists an $\epsilon_d > 0$, such that $M_{F+\epsilon}^X(c_1, d) = \emptyset$ for all d and all $0 < \epsilon \leq \epsilon_d$. As $\langle F, \xi \rangle \neq 0$ for all ξ of type c_1 , we get that for all $d > 0$

$$\text{Coeff}_{\Lambda^d}[\chi_{c_1}^{X,F}(L)] = \chi(M_{F+\epsilon_d}^X(c_1, d), \mu(L)) = 0,$$

and therefore $\chi_{c_1}^{X,F}(L) = 0$.

This result together with Theorem 4.19 implies that for many rational surfaces $\chi_{c_1}^{X,F}(L) = 0$ for all $F \in S_X$.

Theorem 4.22. *Let X be $\mathbb{P}^1 \times \mathbb{P}^1$ or the blowup of \mathbb{P}^2 in at most 7 general points. Let $c_1 \in H^2(X, \mathbb{Z})$ and let L be a line bundle on X with $\langle c_1, L \rangle$ even. Let $F \in S_X$. Then $\chi_{c_1}^{X,F}(L) = 0$.*

Proof. We note that $-K_X$ is ample on X . If there is a morphism $\pi : X \rightarrow \mathbb{P}^1$ with general fibre isomorphic to \mathbb{P}^1 , if G is the class of a fibre of π , then $G \in S_X$. Furthermore, if $\langle c_1, G \rangle$ odd, we get by Remark 4.21 that $\chi_{c_1}^{X,G}(L) = 0$. Then Theorem 4.19 implies that $\chi_{c_1}^{X,F}(L) = 0$ for all $F \in S_X$.

Let \hat{X} be the blowup of X in a general point. We denote by E the exceptional divisor. Then \hat{X} is the blowup of \mathbb{P}^2 in at most 8 general points and $-K_{\hat{X}}$ is ample on \hat{X} . Denote by

H the pullback of the hyperplane class of \mathbb{P}^2 . Then $G := H - E$ is the class of the fibre of a morphism $\pi : \widehat{X} \rightarrow \mathbb{P}^1$, whose general fibre is \mathbb{P}^1 .

If $\langle c_1, G \rangle$ is odd, then $\chi_{c_1}^{\widehat{X}, G}(L) = 0$, and thus, applying Theorem 4.19 again, we get $\chi_{c_1}^{\widehat{X}, F'}(L) = 0$ for all $F' \in S_{\widehat{X}}$. If $F \in S_X$, then also $F \in S_{\widehat{X}}$ and by Proposition 4.9 we get $\chi_{c_1}^{X, F}(L) = \chi_{c_1}^{\widehat{X}, F}(L) = 0$.

If $\langle c_1, G \rangle$ is even, then $\langle c_1 + E, G \rangle$ is odd, and thus $\chi_{c_1+E}^{\widehat{X}, G}(L) = 0$. Again Theorem 4.19 gives that $\chi_{c_1+E}^{\widehat{X}, F}(L) = 0$ for all $F \in S_X$. Thus we get by Proposition 4.9 that $\chi_{c_1}^{X, F}(L) = \frac{1}{\lambda} \chi_{c_1+E}^{\widehat{X}, F}(L) = 0$. \square

4.6. Blowup polynomials and a higher blowup formula. In this section we introduce and study the "blowup polynomials" $R_n(\lambda, x)$, $S_n(\lambda, x)$, which are related to addition formulas for the standard theta functions $\theta_1(z)$ and $\theta_4(z)$. These are related to "higher blowup formulas": if \widehat{X} is the blowup of X in a point and E the exceptional divisor, they relate $\chi_{c_1}^{X, M}(L)$ to $\chi_{c_1}^{\widehat{X}, M}(L - (n-1)E)$ and $\chi_{c_1+E}^{X, M}(L - (n-1)E)$.

Definition 4.23. Define for all $n \in \mathbb{Z}$ rational functions $R_n, S_n \in \mathbb{Q}(\lambda, x)$ by $R_0 = R_1 = 1$, $S_1 = \lambda$, $S_2 = \lambda x$, the recursion relations

$$(4.24) \quad R_{n+1} = \frac{R_n^2 - \lambda^2 S_n^2}{R_{n-1}}, \quad n \geq 1,$$

$$(4.25) \quad S_{n+1} = \frac{S_n^2 - \lambda^2 R_n^2}{S_{n-1}}, \quad n \geq 2.$$

and $R_{-n} = R_n$, $S_{-n} = S_n$. The definition gives

$$\begin{aligned} R_1 &= 1, \quad R_2 = (1 - \lambda^4), \quad R_3 = -\lambda^4 x^2 + (1 - \lambda^4)^2, \quad R_4 = -\lambda^4 x^4 + (1 - \lambda^4)^4, \\ S_1 &= \lambda, \quad S_2 = \lambda x, \quad S_3 = \lambda(x^2 - (1 - \lambda^4)^2), \quad S_4 = \lambda x((1 - \lambda^8)x^2 - 2(1 - \lambda^4)^3). \end{aligned}$$

One can show that the R_i, S_i are polynomials, but we will not need it here. We will want to show that these polynomials are related to the following expressions in theta functions. We put

$$(4.26) \quad \widetilde{R}_n := \frac{\widetilde{\theta}_4(nh)}{\widetilde{\theta}_4(h)^{n^2}}, \quad \widetilde{S}_n := \frac{\widetilde{\theta}_1(nh)}{\widetilde{\theta}_4(h)^{n^2}}.$$

Proposition 4.27. R_n, S_n satisfy

$$(4.28) \quad \widetilde{R}_n = R_n(\Lambda, M), \quad \widetilde{S}_n = S_n(\Lambda, M).$$

Proof. As $\theta_1(h)$ is odd in h and $\theta_4(h)$ is even in h , it follows that $\widetilde{R}_{-n} = \widetilde{R}_n$, $\widetilde{S}_{-n} = -\widetilde{S}_n$, and by definition we get $\widetilde{R}_0 = 1$, $\widetilde{R}_1 = 1$, $\widetilde{S}_1 = \Lambda$. The duplication formula for $\theta_1(g)$ (see [30, §2.1 Ex. 5])

$$\theta_1(2g)\theta_2\theta_3\theta_4 = 2\theta_1(h)\theta_2(h)\theta_3(h)\theta_4(h)$$

gives

$$(4.29) \quad \tilde{S}_2 = \frac{\theta_1(2h)}{\tilde{\theta}_4(h)^4} = \frac{2\tilde{\theta}_1(h)\theta_2(h)\theta_3(h)}{\theta_2\theta_3\tilde{\theta}_4(h)^3} = \Lambda M.$$

The addition formula for $\theta_4(z)$ (see [30, §2.1 Ex. 1])

$$\theta_4(y+z)\theta_4(y-z)\theta_4^2 = \theta_4(y)^2\theta_4(z)^2 - \theta_1(y)^2\theta_1(z)^2$$

applied to $y = nh$, gives

$$(4.30) \quad \begin{aligned} \tilde{R}_{n+1}\tilde{R}_{n-1} &= \frac{\tilde{\theta}_4((n+1)h)}{\tilde{\theta}_4(h)^{(n+1)^2}} \cdot \frac{\tilde{\theta}_4((n-1)h)}{\tilde{\theta}_4(h)^{(n-1)^2}} \\ &= \frac{\tilde{\theta}_4(nh)^2}{\tilde{\theta}_4(h)^{2n^2}} - \frac{\theta_1(nh)^2\theta_1(h)^2}{\tilde{\theta}_4(h)^{2n^2}\theta_4(h)^2} = \tilde{R}_n^2 - \Lambda^2\tilde{S}_n^2, \end{aligned}$$

where in the last step we have used the definition $\Lambda = \frac{\theta_1(h)}{\tilde{\theta}_4(h)}$. Similarly the addition formula for $\theta_1(z)$ (see [30, §2.1 Ex. 1])

$$\theta_1(y+z)\theta_1(y-z)\theta_4^2 = \theta_1(y)^2\theta_4(z)^2 - \theta_4(y)^2\theta_1(z)^2$$

applied to $y = nh$ gives

$$(4.31) \quad \begin{aligned} \tilde{S}_{n+1}\tilde{S}_{n-1} &= \frac{\tilde{\theta}_1((n+1)h)}{\tilde{\theta}_4(h)^{(n+1)^2}} \cdot \frac{\tilde{\theta}_1((n-1)h)}{\tilde{\theta}_4(h)^{(n-1)^2}} \\ &= \frac{\tilde{\theta}_1(nh)^2}{\tilde{\theta}_4(h)^{2n^2}} - \frac{\tilde{\theta}_4(nh)^2\theta_1(h)^2}{\theta_4(h)^2\tilde{\theta}_4(h)^{2n^2}} = \tilde{S}_n^2 - \Lambda^2\tilde{R}_n^2. \end{aligned}$$

This shows the result. \square

We use this result to prove a higher blowup formula. We will use it here for $n = 2$. In a forthcoming paper the first named author will systematically use the higher blowup formulas to study the K -theoretic Donaldson invariants of \mathbb{P}^2 .

Proposition 4.32. *Let X be \mathbb{P}^2 , $\mathbb{P}^1 \times \mathbb{P}^1$ or the blowup of \mathbb{P}^2 in at most 7 points. Let $c_1 \in H^2(X, \mathbb{Z})$ and let L be a line bundle on X with $\langle c_1, L \rangle$ even. Let $F, M \in C_X \cup S_X$.*

- (1) $\Psi_{\hat{X}, c_1}^{F, M}(L - (n-1)E, \Lambda, \tau) = \Psi_{X, c_1}^{F, M}(L, \Lambda, \tau)R_n(\Lambda, M).$
- (2) $\Psi_{\hat{X}, c_1 + E}^{F, M}(L - (n-1)E, \Lambda, \tau) = \Psi_{X, c_1}^{F, M}(L, \Lambda, \tau)S_n(\Lambda, M).$

Proof. (1) Note that $H^2(\hat{X}, \mathbb{Z}) = H^2(X, \mathbb{Z}) + \mathbb{Z}E$ and $L - (n-1)E - K_{\hat{X}} = L - K_X - nE$. Also we have $(-1)^{\langle K_{\hat{X}}, (nE) \rangle} = (-1)^n$, $\langle (nE), (kE) \rangle = -nk$. By definition we get therefore

$$\begin{aligned} \Theta_{\hat{X}, c_1, K_{\hat{X}}}^{F, M}((L - (n-1)E - K_{\hat{X}})z) &= \Theta_{\hat{X}, c_1, K_{\hat{X}}}^{F, M}\left(\frac{1}{2\pi i}(L - K_X)z\right) \sum_{k \in \mathbb{Z}} (-1)^n e^{\pi i \tau n^2} e^{nkz} \\ &= \Theta_{\hat{X}, c_1, K_{\hat{X}}}^{F, M}\left(\frac{1}{2\pi i}(L - K_X)z\right) \theta_4(nz). \end{aligned}$$

We also see that $\sigma(\widehat{X}) = \sigma(X) - 1$ and $(L - K_X - nE)^2 = (L - K_X)^2 - n^2$. Putting this into the definitions of $\Psi_{\widehat{X},c_1}^{F,M}$ and $\Psi_{X,c_1}^{F,M}$, we see that

$$\Psi_{\widehat{X},c_1}^{F,M}(L - (n-1)E, \Lambda, \tau) = \Psi_{X,c_1}^{F,M}(L, \Lambda, \tau) \frac{\theta_4(nh)}{\theta_4(h)^{n^2}} = \Psi_{X,c_1}^{F,M}(L, \Lambda, \tau) R_n(\Lambda, M).$$

(2) The proof is similar. The same argument as above shows that

$$\begin{aligned} \Theta_{\widehat{X},c_1+E,K_{\widehat{X}}}^{F,M}((L - (n-1)E - K_{\widehat{X}})z) &= \Theta_{\widehat{X},c_1,K_{\widehat{X}}}^{F,M}\left(\frac{1}{2\pi i}(L - K_X)z\right) \sum_{k \in \mathbb{Z}} (-1)^n e^{\pi i \tau (n+1/2)^2} e^{(n+1/2)kz} \\ &= \Theta_{\widehat{X},c_1,K_{\widehat{X}}}^{F,M}\left(\frac{1}{2\pi i}(L - K_X)z\right) \theta_1(nz). \end{aligned}$$

Thus the definitions of $\Psi_{\widehat{X},c_1+E}^{F,M}$ and $\Psi_{X,c_1}^{F,M}$ give that

$$\Psi_{\widehat{X},c_1+E}^{F,M}(L - (n-1)E, \Lambda, \tau) = \Psi_{X,c_1}^{F,M}(L, \Lambda, \tau) \frac{\theta_1(nh)}{\theta_4(h)^{n^2}} = \Psi_{X,c_1}^{F,M}(L, \Lambda, \tau) S_n(\Lambda, M).$$

□

Theorem 4.33. *Let X be \mathbb{P}^2 , $\mathbb{P}^1 \times \mathbb{P}^1$ or the blowup of \mathbb{P}^2 in at most 7 points. Let $c_1 \in H^2(X, \mathbb{Z})$ and let L be a line bundle on X with $\langle c_1, L \rangle$ even. Then we have for all $M \in C_X$*

$$\chi_{c_1}^{X,M}(L) = \frac{\chi_{c_1}^{\widehat{X},M}(L - E)}{1 - \Lambda^4}.$$

Proof. First consider the case that X is not \mathbb{P}^2 . Then by Theorem 4.22 there is an $F \in S_X$ with $\chi_{c_1}^{X,F}(L) = 0 = \chi_{c_1}^{\widehat{X},F}(L - E)$. Thus by Corollary 4.6 and Proposition 4.32 we get

$$\begin{aligned} \chi_{c_1}^{\widehat{X},M}(L - E) &= \text{Coeff}_{q^0} [\Psi_{\widehat{X},c_1}^{F,M}(L - E; \Lambda, \tau)] = \text{Coeff}_{q^0} [\Psi_{X,c_1}^{F,M}(L; \Lambda, \tau) R_2(M, \Lambda)] \\ &= \text{Coeff}_{q^0} [\Psi_{X,c_1}^{F,M}(L; \Lambda, \tau)] (1 - \Lambda^4) = \chi_{c_1}^{X,F}(L - E) (1 - \Lambda^4). \end{aligned}$$

Now assume that $X = \mathbb{P}^2$ and $c_1 = H$ is the hyperplane class. Let p_1, p_2 be two different points of \mathbb{P}^2 . For $i = 1, 2$ let X_i be the blowup of \mathbb{P}^2 in p_i with exceptional divisor E_i , and let X be the blowup of \mathbb{P}^2 in p_1 and p_2 . Let $F_1 := H - E_1$, $F_2 := H - E_2$. Then

$$\chi_H^{\mathbb{P}^2,H}(L) = \chi_H^{X_2,H}(L) = \text{Coeff}_{q^0} [\Psi_{X_2,H}^{F_2,H}(L; \Lambda, \tau)] = \frac{1}{1 - \Lambda^4} \text{Coeff}_{q^0} [\Psi_{X,H}^{F_2,H}(L - E_1; \Lambda, \tau)].$$

On the other hand by Theorem 4.19 $\text{Coeff}_{q^0} [\Psi_{X,H}^{F_2,F_1}(L - E_1; \Lambda, \tau)] = 0$, and thus we get by Proposition 4.9

$$\begin{aligned} \text{Coeff}_{q^0} [\Psi_{X,H}^{F_2,H}(L - E_1; \Lambda, \tau)] &= \text{Coeff}_{q^0} [\Psi_{X,H}^{F_1,H}(L - E_1; \Lambda, \tau)] = \text{Coeff}_{q^0} [\Psi_{X_1,H}^{F_1,H}(L - E_1; \Lambda, \tau)] \\ &= \chi_H^{X_1,H}(L - E_1). \end{aligned}$$

Finally let $X = \mathbb{P}^2$ and $c_1 = 0$. We use again Proposition 4.9 and the same argument to get

$$\chi_0^{\mathbb{P}^2,H}(L) = \chi_0^{X_2,H}(L) = \text{Coeff}_{q^0} [\Psi_{X_2,0}^{F_2,H}(nH; \Lambda, \tau)] = \frac{1}{1 - \Lambda^4} (\text{Coeff}_{q^0} [\Psi_{X,0}^{F_2,H}(nH - E_1; \Lambda, \tau)]).$$

Again by Theorem 4.19 we get $\text{Coeff}_{q^0} [\Psi_{X,0}^{F_2,F_1}(nH - E_1; \Lambda, \tau)] = 0$, and thus by Theorem 4.22

$$\begin{aligned} \text{Coeff}_{q^0} [\Psi_{X,0}^{F_2,H}(nH - E_1; \Lambda, \tau)] &= \text{Coeff}_{q^0} [\Psi_{X,0}^{F_1,H}(nH - E_1; \Lambda, \tau)] = \text{Coeff}_{q^0} [\Psi_{X_1,0}^{F_1,H}(nH - E_1; \Lambda, \tau)] \\ &= \chi_0^{X_1,H}(nH - E_1). \end{aligned}$$

The claim follows. \square

5. K -THEORETIC DONALDSON INVARIANTS OF RATIONAL RULED SURFACES

In this section we will compute generating functions for K -theoretic Donaldson invariants of rational ruled surfaces, proving Theorem 1.2. We will do this by proving some recursion formulas for them, which determine them, once suitable initial conditions are satisfied.

5.1. The limit of the invariant at the boundary point. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ or $\widehat{\mathbb{P}}^2$ the blowup of \mathbb{P}^2 in a point. We denote the line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ and $\widehat{\mathbb{P}}^2$ in a uniform way.

Notation 5.1. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ or $X = \widehat{\mathbb{P}}^2$. In the case $X = \mathbb{P}^1 \times \mathbb{P}^1$ we denote F the class of the fibre of the projection to the first factor, and by G the class of the fibre of the projection to the second factor. In the case $X = \widehat{\mathbb{P}}^2$, let H be the pullback of the hyperplane class on \mathbb{P}^2 and E the class of the exceptional divisor. Then $F := H - E$ is the fibre of the ruling of X . We put $G := \frac{1}{2}(H + E)$. Note that G is not an integral cohomology class. In fact, while $H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z}) = \mathbb{Z}F \oplus \mathbb{Z}G$, we have

$$H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z}) = \mathbb{Z}H \oplus \mathbb{Z}E = \{aF + bG \mid a \in \mathbb{Z}, b \in 2\mathbb{Z} \text{ or } a \in \mathbb{Z} + \frac{1}{2}, b \in 2\mathbb{Z} + 1\}.$$

On the other hand we note that both on $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $\widehat{\mathbb{P}}^2$ we have $F^2 = G^2 = 0$, $\langle F, G \rangle = 1$, and $-K_X = 2F + 2G$.

We want to define and study the limit of the K -theoretic Donaldson invariant $\chi(M_P^X(c_1, d), \mu(L))$, as the ample class P tends to F . For $c_1 = F$ or $c_1 = 0$ this will be different from our previous definition of $\chi(M_F^X(c_1, d), \mu(L))$.

Definition 5.2. Fix $d \in \mathbb{Z}$ with $d \equiv -c_1^2(4)$. For $n_d > 0$ sufficiently large, $n_d F + G$ is ample on X , and the condition $\langle c_1, F \rangle$ odd implies that there is no wall ξ of type (c_1, d) with $\langle \xi, (n_d F + G) \rangle > 0 > \langle \xi, F \rangle$. Let $L \in \text{Pic}(X)$ with $\langle c_1, L \rangle$ even. We define

$$\begin{aligned} M_{F_+}^X(c_1, d) &:= M_{n_d F + G}^X(c_1, d), \\ \chi(M_{F_+}^X(c_1, d), \mu(L)) &:= \chi(M_{n_d F + G}^X(c_1, d), \mu(L)), \\ \chi_{c_1}^{X, F_+}(L) &:= \sum_{d \geq 0} \chi(M_{F_+}^X(c_1, d), \mu(L)) \Lambda^d. \end{aligned}$$

Now we give a formula for $\chi_0^{X,F+}(nF + mG)$ and $\chi_F^{X,F+}(nF + mG)$. The rest of this section will be mostly devoted to giving an explicit evaluation of this formula for $m \leq 2$. In work in progress the method will be generalised for higher values of m and also to c_1 different from 0 and F .

Proposition 5.3. *Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ or $X = \widehat{\mathbb{P}}^2$.*

(1) *Let $nF + mG$ be a line bundle on X with m even. Then*

$$\chi_F^{X,F+}(nF + mG) = \text{Coeff}_{q^0} \left[\frac{1}{2 \sinh((m/2 + 1)h)} \Lambda^2 \tilde{\theta}_4(h)^{2(n+2)(m+2)} u' h^* \right].$$

(2) *Let $nF + mG$ be a line bundle on X . Then*

$$\chi_0^{X,F+}(nF + mG) = - \text{Coeff}_{q^0} \left[\frac{1}{2} (\coth((m/2 + 1)h)) \Lambda^2 \tilde{\theta}_4(h)^{2(n+2)(m+2)} u' h^* \right].$$

Proof. We denote $\Gamma_X = H^2(X, \mathbb{Z})$ with inner product the negative of the intersection form. Let $c_1 = 0$ or $c_1 = F$, fix d , and let $s \in \mathbb{Z}_{\geq 0}$ be sufficiently large so that there is no class ξ of (c_1, d) with $\langle \xi, F \rangle < 0 < \langle \xi, G + sF \rangle$. Write $L := nF + mG$. By definition

$$\begin{aligned} \chi(M_X^{sF+G}(F, d), \mu(L)) &= \text{Coeff}_{\Lambda^d} \text{Coeff}_{q^0} \left[\Psi_{X,F}^{F,G+sF}(L; \Lambda, \tau) \right] \\ &= \text{Coeff}_{\Lambda^d} \text{Coeff}_{q^0} \left[\Theta_{\Gamma_X, F, K_X}^{F,G+sF} \left(\frac{1}{2\pi i} (L - K_X)h, \tau \right) \Lambda^2 \tilde{\theta}_4(h)^{(L-K_X)^2} u' h^* \right] \\ &= \text{Coeff}_{\Lambda^d} \text{Coeff}_{q^0} \left[\frac{e^{-\langle \frac{F}{2}, L-K_X \rangle}}{1 - e^{-\langle F, L-K_X \rangle h}} \Lambda^2 \tilde{\theta}_4(h)^{(L-K_X)^2} u' h^* \right] + \sum_{\langle F, \xi \rangle > 0 > \langle G+sF, \xi \rangle} \delta_\xi^X(L). \end{aligned}$$

Here the second sum is over the classes of type (F, d) . By our assumption on n the second sum is empty, so we get

$$\chi_F^{X,F+}(L) = \text{Coeff}_{q^0} \left[\frac{e^{-\langle \frac{F}{2}, L-K_X \rangle}}{1 - e^{-\langle F, L-K_X \rangle h}} \right] = \text{Coeff}_{q^0} \left[\frac{\Lambda^2 \tilde{\theta}_4(h)^{(L-K_X)^2} u' h^*}{2 \sinh(\langle \frac{F}{2}, L - K_X \rangle h)} \right].$$

In the case $c_1 = 0$ the argument is very similar. By definition and Theorem 4.22 we have

$$\begin{aligned} \chi(M_X^{nF+G}(0, d), \mu(L)) &= \text{Coeff}_{\Lambda^d} \text{Coeff}_{q^0} \left[\Psi_{X,0}^{F,sF+G}(L; \Lambda, \tau) \right] \\ &= \text{Coeff}_{\Lambda^d} \text{Coeff}_{q^0} \left[\Theta_{\Gamma_X, 0, K_X}^{F,sF+G} \left(\frac{1}{2\pi i} (L - K_X)h, \tau \right) \Lambda^2 \tilde{\theta}_4(h)^{(L-K_X)^2} u' h^* \right] \\ &= \text{Coeff}_{\Lambda^d} \text{Coeff}_{q^0} \left[\frac{\Lambda^2 \tilde{\theta}_4(h)^{(L-K_X)^2} u' h^*}{1 - e^{-\langle F, L-K_X \rangle h}} \right] + \sum_{\langle F, \xi \rangle > 0 > \langle M, \xi \rangle} \delta_\xi^X(L). \end{aligned}$$

The second sum is again over the walls of type $(0, d)$, and thus it is 0. Thus we get

$$\begin{aligned} \chi_0^{X,F+}(L) &= \text{Coeff}_{q^0} \left[\frac{\Lambda^2 \tilde{\theta}_4(h)^{(L-K_X)^2} u' h^*}{1 - e^{-\langle F, L-K_X \rangle h}} \right] \\ &= - \text{Coeff}_{q^0} \left[\frac{1}{2} (\coth(\langle F, (L - K_X)/2 \rangle h) + 1) \Lambda^2 \tilde{\theta}_4(h)^{(L-K_X)^2} u' h^* \right]. \end{aligned}$$

Note that by Remark 3.15, we get

$$\text{Coeff}_{q^0}[\Lambda^2 \tilde{\theta}_4(h)^{(L-K_X)^2} u' h^*] = \text{Coeff}_{q^0}[(1-1) \Lambda^2 \tilde{\theta}_4(h)^{(L-K_X)^2} u' h^*] = 0.$$

□

Remark 5.4. In the case $\mathbb{P}^1 \times \mathbb{P}^1$, we can in the same way define $M_{G_+}^{\mathbb{P}^1 \times \mathbb{P}^1}(c_1, d) := M_{n_d G_+ + F}^{\mathbb{P}^1 \times \mathbb{P}^1}(c_1, d)$ for n_d sufficiently large with respect to d , and

$$\chi_{c_1}^{\mathbb{P}^1 \times \mathbb{P}^1, G_+}(nF + mG) := \sum_{d > 0} \chi(M_{G_+}^{\mathbb{P}^1 \times \mathbb{P}^1}(c_1, d), \mu(nF + mG)).$$

Then we see immediately that $\chi_F^{\mathbb{P}^1 \times \mathbb{P}^1, G_+}(nF + mG) = 0$, and we get by symmetry from Proposition 5.3 that

$$\chi_0^{\mathbb{P}^1 \times \mathbb{P}^1, G_+}(nF + mG) = -\text{Coeff}_{q^0} \left[\frac{1}{2} (\coth((n/2 + 1)h)) \Lambda^2 \tilde{\theta}_4(h)^{2(n+2)(m+2)} u' h^* \right].$$

5.2. Theta constant identities. We use the blowup polynomials $R_n(x, \lambda)$, $S_n(x, \lambda)$ and the blowup functions $\tilde{R}_n = R_n(M, \Lambda)$, $\tilde{S}_n = S_n(M, \Lambda)$ of Definition 4.23 to find identities between expressions in theta functions, evaluated at some division point. These will then below be used to give recursion formulas for K -theoretic Donaldson invariants of rational ruled surfaces.

Definition 5.5. Fix $\tau \in \mathcal{H}$. Let $r \in \mathbb{Z}_{\geq 0}$ and $l \in \mathbb{Z}$ with $l \equiv r(2)$.

$$\begin{aligned} RR_{\frac{r-l}{2}, \frac{r+l}{2}}(h) &:= \tilde{R}_{\frac{r-l}{2}}(h) - \tilde{\theta}_4(z)^{rl} \tilde{R}_{\frac{r+l}{2}}(h), \\ SS_{\frac{r-l}{2}, \frac{r+l}{2}}(h) &:= \tilde{S}_{\frac{r-l}{2}}(h) + e^{\pi i r \Re(h)} \tilde{\theta}_4(h)^{rl} \tilde{S}_{\frac{r+l}{2}}(h). \end{aligned}$$

Proposition 5.6. Fix $r \in \mathbb{Z}_{\geq 0}$, and fix $a \in \mathbb{Z}$. Let $l \in \mathbb{Z}$ with $l \equiv r(2)$. Then

- (1) $RR_{\frac{r-l}{2}, \frac{r+l}{2}}(2\pi i \frac{a}{r}) = 0$,
- (2) $SS_{\frac{r-l}{2}, \frac{r+l}{2}}(2\pi i \frac{a}{r}) = 0$.

Proof. (1) Put $z_0 := 2\pi i \frac{a}{r}$. As $\theta_4(h)$ is even, we get

$$\theta_4\left(\frac{r-l}{2}z_0\right) = \theta_4\left(-\frac{r-l}{2}z_0\right) = \theta_4\left(-2\pi i \frac{r-l}{2r}a\right) = \theta_4\left(2\pi i \frac{r+l}{2r}a\right) = \theta_4\left(\frac{r+l}{2}z_0\right),$$

where we used $\theta_4(z+a) = \theta_4(z)$. By the definition of \tilde{R}_n we get

$$\tilde{R}_{\frac{r-l}{2}}(z_0) = \frac{\tilde{\theta}_4(\frac{r-l}{2}z_0)}{\tilde{\theta}_4(z_0)^{\frac{(r-l)^2}{4}}} = \frac{\tilde{\theta}_4(\frac{r+l}{2}z_0)}{\tilde{\theta}_4(z_0)^{\frac{(r-l)^2}{4}}} = \tilde{\theta}_4(z_0)^{rl} \frac{\tilde{\theta}_4(\frac{r+l}{2}z_0)}{\tilde{\theta}_4(z_0)^{\frac{(r+l)^2}{4}}} = \tilde{\theta}_4(z_0)^{rl} \tilde{R}_{\frac{r+l}{2}}(z_0).$$

(2) As $\theta_1(z)$ is odd, and $\theta_1(z + 2\pi i a) = (-1)^a \theta_1(z)$, we get

$$\theta_1\left(\frac{r-l}{2}z_0\right) = -\theta_1\left(-\frac{r-l}{2}z_0\right) = -\theta_1\left(-2\pi i \frac{r-l}{2r}a\right) = (-1)^{a+1} \theta_1\left(2\pi i \frac{r+l}{2r}a\right) = (-1)^{a+1} \theta_1\left(\frac{r+l}{2}z_0\right).$$

By the definition of \tilde{S}_n we get

$$\tilde{S}_{\frac{r-l}{2}}(z_0) = \frac{\tilde{\theta}_1(\frac{r-l}{2}z_0)}{\tilde{\theta}_4(z_0)^{\frac{(r-l)^2}{4}}} = (-1)^{a+1}\tilde{\theta}_4(z_0)^{rl} \frac{\tilde{\theta}_1(\frac{r+l}{2}z_0)}{\tilde{\theta}_4(z_0)^{\frac{(r+l)^2}{4}}} = (-1)^{a+1}\tilde{\theta}_4(z_0)^{rl}\tilde{S}_{\frac{r+l}{2}}(z_0).$$

□

Proposition 5.7. *Let $a \in \mathbb{Z}$.*

- (1) Put $Q_2(z) := \tilde{\theta}_4(z)^4(1 - \Lambda^4)$. Then $(Q_2(z) - 1)|_{z=\pi ia} = 0$.
- (2) Put $Q_3(z) := \tilde{\theta}_4(z)^3(1 - \Lambda^4)$. Then $(Q_3(z) - 1)|_{z=2\pi i\frac{a}{3}} = 0$.
- (3) Put $Q_4(z) := \tilde{\theta}_4(z)^8(1 - \Lambda^4)^3$. Then $(Q_4(z) - (1 - (-1)^a\Lambda^4))|_{z=\pi i\frac{a}{2}} = 0$.

Proof. (1) Put $z_0 = \pi ia$. Applying (1) of Proposition 5.6, with $r = 2$, $l = 2$, we get $\tilde{R}_0(z_0) = \tilde{\theta}_4(z_0)^4\tilde{R}_2(z_0)$, and the claim follows because $\tilde{R}_0 = 1$, $\tilde{R}_2 = (1 - \Lambda^4)$.

(2) Put $z_0 = 2\pi i\frac{a}{3}$. Applying (1) of Proposition 5.6, with $r = 3$, $l = 1$, we get $\tilde{R}_1(z_0) = \tilde{\theta}_4(z_0)^3\tilde{R}_2(z_0)$, and the claim follows because $\tilde{R}_1 = 1$, $\tilde{R}_2 = (1 - \Lambda^4)$.

(3) Put $z_0 = \pi i\frac{a}{2}$. Apply (1) and (2) of Proposition 5.6, with $r = 4$, $l = 2$. This gives

$$(5.8) \quad 1 = \tilde{R}_1(z_0) = \tilde{\theta}_4(z)^8\tilde{R}_3(z_0),$$

$$(5.9) \quad 1 = \tilde{S}_1(z_0) = -(-1)^a\tilde{\theta}_4(z)^8\tilde{S}_3(z_0).$$

We have $\tilde{R}_3 = -\Lambda^4 M^2 + (1 - \Lambda^4)^2$, $\tilde{S}_3 = M^2 - (1 - \Lambda^4)^2$. Thus subtracting $(-1)^a\Lambda^4(z_0)$ times (5.9) from (5.8) we get

$$(1 - (-1)^a\Lambda(z_0)^4) = \tilde{\theta}_4(z)^8(1 - \Lambda(z_0)^4)^3.$$

□

5.3. Recursion formulas from theta constant identities. We now use the theta constant identities of Proposition 5.6 to show recursion formulas in n for the K -theoretical Donaldson invariants $\chi_0^{X,F+}(nF + mG)$, $\chi_F^{X,F+}(nF + mG)$ for $0 \leq m \leq 2$ for polarizations near the fibre class. We consider expressions relating the left hand sides of the formulas of Proposition 5.3 for $\chi_0^{X,F+}(nF + mG)$, $\chi_F^{X,F+}(nF + mG)$ for successive values of n . We show that the theta constant identities of Proposition 5.6 imply that these expressions are almost holomorphic in q , i.e. that they have only finitely many monomials $\Lambda^d q^s$ with nonzero coefficients and $s \leq 0$. This will then give recursion formulas for $\chi_0^{X,F+}(nF + mG)$, $\chi_F^{X,F+}(nF + mG)$.

Proposition 5.10.

$$\begin{aligned}
(1) \quad & \tilde{\theta}_4(h) = 1 + q^2\Lambda^2 + O(q^3), \\
(2) \quad & -\frac{1}{2} \coth(h/2) u' h^* \Lambda^2 = -\frac{1}{2} q^{-2} \Lambda^2 - \Lambda^4 + O(q), \\
(3) \quad & \frac{1}{2 \sinh(h)} \left(\tilde{\theta}_4(h)^4 (1 - \Lambda^4) - 1 \right) u' h^* \Lambda^2 = \Lambda^4 + O(q), \\
(4) \quad & -\frac{1}{2} \coth(h) \left(\tilde{\theta}_4(h)^4 (1 - \Lambda^4) - 1 \right) u' h^* \Lambda^2 = -\Lambda^4 + \frac{1}{2} q^{-2} \Lambda^6 + 3\Lambda^8 + O(q), \\
(5) \quad & -\frac{1}{2} \coth(3h/2) \left(\tilde{\theta}_4(h)^3 (1 - \Lambda^4) - 1 \right) u' h^* \Lambda^2 = -\frac{1}{2} \Lambda^4 + \frac{1}{2} q^{-2} \Lambda^6 + \frac{5}{2} \Lambda^8 + O(q), \\
(6) \quad & -\frac{1}{2} \tanh(h) \left(\tilde{\theta}_4(h)^8 (1 - \Lambda^4)^3 - (1 + \Lambda^4) \right) u' h^* \Lambda^2 \\
& = 2q^{-2} \Lambda^6 + 13\Lambda^8 - \frac{3}{2} q^{-2} \Lambda^{10} - 14\Lambda^{12} + \frac{1}{2} q^{-2} \Lambda^{14} + 5\Lambda^{16} + O(q).
\end{aligned}$$

Proof. (1) was already shown in Lemma 3.18(4).

(2) In Lemma 3.18 it was shown that $\frac{1}{\sinh(h/2)} \in q\Lambda^{-1}\mathcal{R}$, and that $\cosh(h/2)h^* \in q^{-1}\Lambda\mathcal{R}$. It is also easy to see that the coefficient of $q\Lambda^{-1}$ of $\frac{1}{\sinh(h/2)}$ is $-2i$. As $u \in q^{-2}\mathbb{Q}[[q]]$, we find therefore that

$$\frac{\cosh(h/2)}{2 \sinh(h/2)} u' h^* \Lambda^2 \in q^{-2} \Lambda^2 \mathcal{R},$$

and explicit computation with lower order coefficients of the power series involved determines the coefficients of degree at most 0 in q . This shows (2).

By definition it is easy to see that $\tilde{\theta}_1(h) \in (y - y^{-1})q\mathbb{Q}[[y^{\pm 2}q^4, q^4]]$, and we get

$$(5.11) \quad \tilde{\theta}_4(h) \in \mathbb{Q}[[y^{\pm 2}q^4, q^4]], \quad \Lambda^4 \in \mathbb{Q}[y^2, y^{-2}]_{\leq 1} \mathbb{Q}[[y^{\pm 2}q^4, q^4]].$$

(3) Let

$$f(y, q) = \sum_{n \geq 0} f_n(y) q^{4n} = \tilde{\theta}_4(h)^4 (1 - \Lambda^4) - 1.$$

By the above $f(y, q) \in \mathbb{Q}[y^2, y^{-2}]_{\leq 1} \mathbb{Q}[[y^{\pm 2}q^4, q^4]]$. Furthermore $f(y, q)$ is symmetric under $y \leftrightarrow y^{-1}$. By Proposition 5.7(1) the function $f(y, q)$ vanishes (identically in q) for $2h \in 2\pi i\mathbb{Z}$, i.e. if $y^4 = 1$, i.e. for $y^2 = y^{-2}$. Thus every coefficient $f_n(y)$ is as a symmetric Laurent polynomial in y^2 divisible by $y^2 - y^{-2}$. Let

$$g(y, q) = \sum_{n \geq 1} g_n(y) q^{4n} := \frac{1}{2 \sinh(h)} (\tilde{\theta}_4(h)^4 (1 - \Lambda^4) - 1).$$

Then $g_1 = -(y^2 - y^{-2})$ and g_n an antisymmetric Laurent polynomial in y^2 of degree at most n . Thus for all n we get that $g_n(y)$ is a \mathbb{Q} -linear combination of $\sinh(kh)$ with $k = 1, \dots, n$. Therefore we get by Lemma 3.18 that

$$(5.12) \quad g_n(y) h^* u' \Lambda^2 \in \mathbb{Q}[q^{-2} \Lambda^2]_{\leq n+1} \mathcal{R}.$$

Thus the only possible monomial $\Lambda^m q^n$ with non vanishing coefficient of

$$g(y, q)u'h^*\Lambda^2 = \sum_n g_n(y)h^*u'\Lambda^2 q^{4n},$$

with nonpositive power of q is $\Lambda^4 q^0$, and direct simple computation gives that its coefficient is 1. This shows (3)

(4) Note that

$$-\frac{1}{2} \coth(h)(\tilde{\theta}_4(h)^4(1 - \Lambda^4) - 1) = -\cosh(h)g(y, q).$$

By Lemma 3.18 we have $\cosh(h) \in 1 + (q^{-2}\Lambda^2)_{\leq 1}\mathcal{R}$. By (5.12) this implies that the only monomials in Λ, q with non vanishing coefficients and nonpositive degree in q in $-\cosh(h)g(y, q)$ are $\Lambda^4 q^0, \Lambda^6 q^2, \Lambda^8 q^0$. A simple direct computation with the leading terms of the Laurent series involved determines the coefficients of these monomials.

(5) Let

$$f^1(y, q) = \sum_{n>0} f_n^1(y)q^{4n} = \tilde{\theta}_4(h)^3(1 - \Lambda^4) - 1.$$

By (5.11) we get that $f^1(y, q) \in \mathbb{Q}[y^2, y^{-2}]_{\leq 1}\mathbb{Q}[[y^{\pm 2}q^4, q^4]]$. Again we see that every $f_n^1(y)$ is a symmetric Laurent polynomial in y^2 . By Proposition 5.7 we have that $f^1(y, q)$ vanishes (identically in q) for $3h \in 2\pi i\mathbb{Z}$, i.e. for $y^3 = y^{-3}$. Thus every coefficient $f_n^1(y)$ is a symmetric Laurent polynomial in y divisible by $y^3 - y^{-3}$. Let

$$(5.13) \quad g^1(y, q) = \sum_n g_n^1(y)q^{4n} := \frac{1}{2} \coth(3h/2)f^1(y, q)$$

Then all $g_n^1(y) = \frac{y^3 + y^{-3}}{2(y^3 - y^{-3})}f_n^1(y)$ are antisymmetric Laurent polynomials in y^2 of degree at most $n + 1$. Thus all $g_n^1(y)$ are linear combinations of $\sinh(kh)$ for $k = 1, \dots, n + 1$. Therefore we get by Lemma 3.18 that

$$g_n^1(y)h^*u'\Lambda^2 \in \mathbb{Q}[q^{-2}\Lambda^2]_{\leq n+2}\mathbb{Q}[[\Lambda^2 q^2, q^4]].$$

Thus we get by (5.13) that the only monomials $\Lambda^m q^n$ with non vanishing coefficients in $g^1(y, q)u'h^*\Lambda^2$ are $\Lambda^4, q^{-2}\Lambda^6, \Lambda^8$, and their coefficients are again determined by a simple direct computation.

(6) Let

$$f^2(y, q) = \sum_{n>0} f_n^2(y)q^{4n} = \tilde{\theta}_4(h)^8(1 - \Lambda^4)^3 - (1 + \Lambda^4).$$

By (5.11) we get that $f^2(y, q) \in \mathbb{Q}[y^2, y^{-2}]_{\leq 3}\mathbb{Q}[[y^{\pm 2}q^4, q^4]]$. Thus again every coefficient $f_n^2(y)$ is a symmetric Laurent polynomial in y^2 . By Proposition 5.7 $f^2(y, q)$ vanishes (identically in q) for $4h \in 2\pi i\mathbb{Z}$, but $2h \notin 2\pi i\mathbb{Z}$. Thus every $f_n^2(y)$ is a symmetric Laurent polynomial in y^2 divisible by $\frac{y^4 - y^{-4}}{y^2 - y^{-2}} = y^2 + y^{-2}$. Let

$$g^2(y, q) := \sum_n g_n^2(y)q^{4n} := \frac{1}{2} \tanh(h)f^2(y, q).$$

Then all $g_n^2(y)$ are antisymmetric Laurent polynomials in y^2 of degree at most $n + 3$. Thus $g_n^2(y)h^*u'\Lambda^2 \in \mathbb{Q}[q^{-2}\Lambda^2]_{\leq n+4}\mathcal{R}$. By the definition of $g^2(y, q)$ this gives that the only monomials $\Lambda^m q^n$ with non vanishing coefficients in $g^2(y, q)u'h^*\Lambda^2$ are Λ^4 , $q^{-2}\Lambda^6$, $q^{-4}\Lambda^8$, Λ^8 , $q^{-6}\Lambda^{10}$, $q^{-2}\Lambda^{10}$, $q^{-4}\Lambda^{12}$, Λ^{12} , $q^{-2}\Lambda^{14}$, Λ^{16} , and their coefficients are again determined by a simple direct computation. \square

Proposition 5.14. *For $X = \mathbb{P}^1 \times \mathbb{P}^1$ or $X = \widehat{\mathbb{P}}^2$ and all $n \in \mathbb{Z}$ we have*

$$(1) \quad 1 + \chi_F^{X, F_+}(nF) = \frac{1}{(1 - \Lambda^4)^{n+1}},$$

$$(2) \quad 1 + (2n + 5)\Lambda^4 + \chi_0^{X, F_+}(nF) = \frac{1}{(1 - \Lambda^4)^{n+1}}.$$

Proof. (1) By Proposition 5.3 we need to show

$$(5.15) \quad 1 + \text{Coeff}_{q^0} \left[\frac{1}{2 \sinh(h)} \left(\frac{\theta_4(h)}{\theta_4} \right)^{4(n+2)} u'h^*\Lambda^2 \right] = \frac{1}{(1 - \Lambda^4)^{n+1}}.$$

The proof is by both descending and ascending induction on $n \in \mathbb{Z}$. We first study the case $n = -1$.

By Proposition 5.3 we know that

$$\chi_F^{\mathbb{P}^1 \times \mathbb{P}^1, F_+}(-F) = \text{Coeff}_{q^0} \left[\frac{1}{2 \sinh(h)} \Lambda^2 \tilde{\theta}_4^4 u'h^* \right].$$

On the other hand $\chi_F^{\mathbb{P}^1 \times \mathbb{P}^1, G}(-F) = 0$ and thus

$$\chi_F^{\mathbb{P}^1 \times \mathbb{P}^1, F_+}(-F) = \sum_{\langle \xi, F \rangle > 0 > \langle \xi, G \rangle} \delta_\xi^{\mathbb{P}^1 \times \mathbb{P}^1}(-F),$$

where the sum is over classes of type (F) , i.e. over all $\xi = -(2n-1)F + 2mG$ with $n, m \in \mathbb{Z}_{\geq 0}$. By Theorem 3.19 we get that $\delta_{-(2n-1)F+2mG}^{\mathbb{P}^1 \times \mathbb{P}^1}(-F) = 0$ unless $8nm - 4m \leq |4n - 2 - 2m| + 2$. This means in case $4n - 2 - 2m \geq 0$ that $8nm - 4m \leq 4n - 2m$, which is impossible, and in case $4n - 2 - 2m \leq 0$ that $8nm - 4m \leq 2m - 4n + 4$, which is also impossible. Thus $\chi_F^{\mathbb{P}^1 \times \mathbb{P}^1, F_+}(-F) = 0$ and therefore also $\text{Coeff}_{q^0} \left[\frac{1}{2 \sinh(h)} \Lambda^2 \tilde{\theta}_4^4 u'h^* \right] = 0$. This shows the case $n = -1$.

By Lemma 3.18 we have $\tilde{\theta}_4(h) = 1 + O(q)$. Thus we get by Proposition 5.10(3), for all $n \in \mathbb{Z}$ that

$$\begin{aligned} & \frac{1}{2 \sinh(h)} (\tilde{\theta}_4(h)^{4(n+2)} (1 - \Lambda^4) - \tilde{\theta}_4(h)^{4(n+1)}) u'h^*\Lambda^2 \\ &= \tilde{\theta}_4(h)^{4(n+1)} \frac{1}{2 \sinh(h)} (\tilde{\theta}_4(h)^4 (1 - \Lambda^4) - 1) u'h^*\Lambda^2 = \Lambda^4 + O(q). \end{aligned}$$

Thus, writing

$$f_n := 1 + \text{Coeff}_{q^0} \left[\frac{1}{2 \sinh(h)} \tilde{\theta}_4(h)^{4(n+2)} u'h^*\Lambda^2 \right],$$

we have $(1 - \Lambda^4)f_n - f_{n-1} = \Lambda^4 + (1 - \Lambda^4) - 1 = 0$, i.e. $f_n(1 - \Lambda^4) = f_{n-1}$ for all $n \in \mathbb{Z}$. We have shown above that $f_{-1} = 1$. Thus the claim follows.

(2) By Proposition 5.3 we know that

$$\chi_0^{\mathbb{P}^1 \times \mathbb{P}^1, F_+}(-F) = \text{Coeff}_{q^0} \left[-\frac{1}{2} \coth(h) \Lambda^2 \tilde{\theta}_4^4 u' h^* \right].$$

By Proposition 5.10(1) we have $\tilde{\theta}_4(h) = 1 + q^2 \Lambda^2 + O(q^3)$. Thus Proposition 5.10(2) gives $\text{Coeff}_{q^0} \left[-\frac{1}{2} \coth(h/2) \Lambda^2 \tilde{\theta}_4^4 u' h^* \right] = -3\Lambda^4$. Thus we get by Remark 5.4 that

$$\chi_0^{\mathbb{P}^1 \times \mathbb{P}^1, G_+}(-F) = \text{Coeff}_{q^0} \left[-\frac{1}{2} \coth(h/2) \Lambda^2 \tilde{\theta}_4^4 u' h^* \right] = -3\Lambda^4.$$

This gives

$$\chi_0^{\mathbb{P}^1 \times \mathbb{P}^1, F_+}(-F) = \chi_0^{\mathbb{P}^1 \times \mathbb{P}^1, F_+}(-F) - \chi_0^{\mathbb{P}^1 \times \mathbb{P}^1, G_+}(-F) - 3\Lambda^4 = \sum_{\langle \xi, F \rangle > 0 > \langle \xi, G \rangle} \delta_{\xi}^{\mathbb{P}^1 \times \mathbb{P}^1}(-F) - 3\Lambda^4,$$

where the sum is over classes of type (0), i.e. over all $\xi = -2nF + 2mG$ with $n, m \in \mathbb{Z}_{\geq 0}$. By Theorem 3.19 we get that $\delta_{-2nF+2mG}^{\mathbb{P}^1 \times \mathbb{P}^1}(-F) = 0$ unless $8nm \leq |4n - 2m| + 2$, which is impossible.

Thus $\chi_{\mathbb{P}^1 \times \mathbb{P}^1, F}^{F_+}(-F) = -3\Lambda^4$ and therefore also $\text{Coeff}_{q^0} \left[-\frac{1}{2} \coth(h) \Lambda^2 \tilde{\theta}_4^4 u' h^* \right] = -3\Lambda^4$.

$\tilde{\theta}_4(h) = 1 + q^2 \Lambda^2 + O(q^3)$. Thus we get by Proposition 5.10(2), for all $n \in \mathbb{Z}$ that

$$\begin{aligned} & -\frac{1}{2} \coth(h) (\tilde{\theta}_4(h)^{4(n+2)} (1 - \Lambda^4) - \tilde{\theta}_4(h)^{4(n+1)}) u' h^* \Lambda^2 \\ &= -\tilde{\theta}_4(h)^{4(n+1)} \frac{1}{2} \coth(h) (\tilde{\theta}_4(h)^4 (1 - \Lambda^4) - 1) u' h^* \Lambda^2 = \Lambda^4 + (2n + 5) \Lambda^8 + O(q). \end{aligned}$$

We put $g_n := 1 + (2n + 5) \Lambda^4 + \chi_0^{X, F_+}(nF)$. Then we have by Proposition 5.3 that

$$g_n = 1 + (2n + 5) \Lambda^4 + \text{Coeff}_{q^0} \left[-\frac{1}{2} \coth(h) \tilde{\theta}_4(h)^{4(n+2)} u' h^* \Lambda^2 \right].$$

We have by the above $g_{-1} = 1$. We get for all $n \in \mathbb{Z}$ that

$$(1 - \Lambda^4) g_n - g_{n-1} = (1 + (2n + 5) \Lambda^4) (1 - \Lambda^4) - (1 + (2n + 3) \Lambda^4) - \Lambda^4 + (2n + 5) \Lambda^8 = 0,$$

i.e. $(1 - \Lambda^4) g_n = g_{n-1}$. By induction over $n \in \mathbb{Z}$ this gives $g_n = \frac{1}{(1 - \Lambda^4)^{n+1}}$. \square

Proposition 5.16. *For $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $n \in \mathbb{Z}$, and for $X = \widehat{\mathbb{P}}^2$ and $n \in \mathbb{Z} + \frac{1}{2}$ we have*

$$1 + (3n + 7) \Lambda^4 + \chi_0^{X, F_+}(nF + G) = \frac{1}{(1 - \Lambda^4)^{2n+2}}.$$

Proof. We will treat the cases of $\mathbb{P}^1 \times \mathbb{P}^1$ and $\widehat{\mathbb{P}}^2$ together and prove the result by induction over $n \in \frac{1}{2}\mathbb{Z}$. By Proposition 5.3 we have for $n \in \frac{1}{2}\mathbb{Z}$ that

$$\chi_0^{X, F_+}(nF + G) = \text{Coeff}_{q^0} \left[-\frac{1}{2} \coth(3h/2) \tilde{\theta}_4(h)^{6(n+2)} u' h^* \Lambda^2 \right].$$

Here $X = \mathbb{P}^1 \times \mathbb{P}^1$ if $n \in \mathbb{Z}$ and $\widehat{\mathbb{P}}^2$ otherwise. For $n \in \frac{1}{2}\mathbb{Z}$ let

$$h_n := 1 + (3n + 7) \Lambda^4 + \chi_0^{X, F_+}(nF + G) = \text{Coeff}_{q^0} \left[-\frac{1}{2} \coth(3h/2) \tilde{\theta}_4(h)^{6(n+2)} u' h^* \Lambda^2 \right] + 1 + (3n + 7) \Lambda^4.$$

We want to show by induction on $n \in \frac{1}{2}\mathbb{Z}$ that $h_n = \frac{1}{(1-\Lambda^4)^{2n+2}}$.

Case $n = 0$. By Proposition 5.14 and symmetry, we have $1 + 7\Lambda^4 + \chi_0^{\mathbb{P}^1 \times \mathbb{P}^1, G_+}(G) = \frac{1}{(1-\Lambda^4)^2}$. Thus

$$1 + 7\Lambda^4 + \chi_0^{\mathbb{P}^1 \times \mathbb{P}^1, F_+}(G) = \frac{1}{(1-\Lambda^4)^2} + \sum_{\xi} \delta_{\xi}^{\mathbb{P}^1 \times \mathbb{P}^1}(G),$$

where ξ runs over all $-2nF + 2mG$ with $n, m \in \mathbb{Z}_{>0}$. By Theorem 3.19 we have that $\delta_{-2nF+2mG}^{\mathbb{P}^1 \times \mathbb{P}^1}(G) = 0$, unless $8nm \leq |4m - 6n| + 2$, which is impossible. Thus

$$g_0 = 1 + 7\Lambda^4 + \chi_0^{\mathbb{P}^1 \times \mathbb{P}^1, G_+}(G) = \frac{1}{(1-\Lambda^4)^2}.$$

Induction step. By Proposition 5.10 we have

$$-\frac{1}{2} \coth(3h/2) \left(\tilde{\theta}_4(h)^3 (1 - \Lambda^4) - 1 \right) u' h^* \Lambda^2 = -\frac{1}{2} \Lambda^4 + \frac{1}{2} q^{-2} \Lambda^6 + \frac{5}{2} \Lambda^8 + O(q).$$

Using also $\tilde{\theta}_4(h) = 1 + q^2 \Lambda^2 + O(q^3)$ we get

$$-\text{Coeff}_{q^0} \left[\frac{1}{2} \coth(3h/2) \left(\tilde{\theta}_4(h)^{6(n+2)} (1 - \Lambda^4) - \tilde{\theta}_4(h)^{6(n+3/2)} \right) u' h^* \Lambda^2 \right] = 1/2 \Lambda^4 + (3n+7) \Lambda^8.$$

Thus we get

$$h_n(1 - \Lambda^4) - h_{n-1/2} = \frac{1}{2} \Lambda^4 + (3n+7) \Lambda^8 + (1 + (3n+7) \Lambda^4)(1 - \Lambda^4) - (1 + (3n + \frac{11}{2}) \Lambda^4) \Lambda^4 = 0.$$

Thus by induction $h_n = \frac{1}{(1-\Lambda^4)^{2n+2}}$. \square

Proposition 5.17. *Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ or $X = \widehat{\mathbb{P}}^2$.*

(1) *For all $n \in \mathbb{Z}$*

$$\sum_d \chi(M_{F_+}^X(F, d), nF + 2G) \Lambda^d = \frac{1}{2} \frac{(1 + \Lambda^4)^n - (1 - \Lambda^4)^n}{(1 - \Lambda^4)^{3n+3}}.$$

(2) *For all $n \in \mathbb{Z}$:*

$$1 + (4n+9) \Lambda^4 + \sum_d \chi(M_{F_+}^X(0, d), nF + 2G) \Lambda^d = \frac{1}{2} \frac{(1 + \Lambda^4)^n + (1 - \Lambda^4)^n}{(1 - \Lambda^4)^{3n+3}}.$$

Proof. (1) By Proposition 5.3, we have

$$\chi_F^{X, F_+}(nF + 2G) = \text{Coeff}_{q^0} \left[\frac{1}{2 \sinh(2h)} \tilde{\theta}_4(h)^{8(n+2)} u' h^* \Lambda^2 \right].$$

Note that $\frac{1}{2 \sinh(2h)} = \frac{1}{4 \sinh(h) \cosh(h)} = \frac{1}{4} (\coth(h) - \tanh(h))$. By Proposition 5.3 and Proposition 5.14, we have

$$\begin{aligned} \text{Coeff}_{q^0} \left[\frac{1}{2} \coth(h) \tilde{\theta}_4(h)^{8(n+2)} u' h^* \Lambda^2 \right] &= -\chi_0^{X, F_+}((2n+2)F) \\ (5.18) \quad &= -\frac{1}{(1 - \Lambda^4)^{2n+3}} + (1 + (4n+9) \Lambda^4). \end{aligned}$$

We will show by induction on $n \in \mathbb{Z}$ that

$$(5.19) \quad l_n := 1 + (4n + 9)\Lambda^4 - \text{Coeff}_{q^0} \left[\frac{1}{2} \tanh(h) \tilde{\theta}_4(h)^{8(n+2)} u' h^* \Lambda^2 \right] = \frac{(1 + \Lambda^4)^n}{(1 - \Lambda^4)^{3n+3}}.$$

(1) follows directly from (5.18) and (5.19):

$$\begin{aligned} \chi_F^{X, F+}(nF + 2G) &= \text{Coeff}_{q^0} \frac{1}{4} \left(\coth(h) - \tanh(h) \right) \tilde{\theta}_4(h)^{8(n+2)} u' h^* \Lambda^2 \Big] \\ &= \frac{1}{2} \left(-\frac{1}{(1 - \Lambda^4)^{2n+3}} + 1 + (4n + 9)\Lambda^4 + \frac{(1 + \Lambda^4)^n}{(1 - \Lambda^4)^{3n+3}} - 1 - (4n + 9)\Lambda^4 \right) = \frac{1}{2} \frac{(1 + \Lambda^4)^n - (1 - \Lambda^4)^n}{(1 - \Lambda^4)^{3n+3}}. \end{aligned}$$

Case $n = 0$. We have $\chi_F^{\mathbb{P}^1 \times \mathbb{P}^1, G+}(2G) = 0$. Thus

$$\chi_F^{\mathbb{P}^1 \times \mathbb{P}^1, F+}(2G) = \sum_{\xi} \delta_{\xi}^{\mathbb{P}^1 \times \mathbb{P}^1}(2G),$$

where ξ runs over all $-(2n - 1)F + 2mG$ with $n, m \in \mathbb{Z}_{>0}$. By Theorem 3.19 we have that $\delta_{-(2n-1)F+2mG}^{\mathbb{P}^1 \times \mathbb{P}^1}(G) = 0$, unless $8nm - 4m \leq |4m - 8n + 4| + 2$, which is impossible. Thus by the above

$$0 = \chi_F^{\mathbb{P}^1 \times \mathbb{P}^1, F+}(2G) = \text{Coeff}_{q^0} \left[\frac{1}{4} (\coth(h) - \tanh(h)) \tilde{\theta}_4(h)^{16} u' h^* \Lambda^2 \right].$$

This gives

$$-\text{Coeff}_{q^0} \left[\frac{1}{2} \tanh(h) \tilde{\theta}_4(h)^{16} u' h^* \Lambda^2 \right] = -\text{Coeff}_{q^0} \left[\frac{1}{2} \coth(h) \tilde{\theta}_4(h)^{16} u' h^* \Lambda^2 \right] = \frac{1}{(1 - \Lambda)^3} - 1 - 9\Lambda^4.$$

Induction step. By Proposition 5.10 we have

$$\begin{aligned} &-\frac{1}{2} \tanh(h) (\tilde{\theta}_4(h)^8 (1 - \Lambda^4)^3 - (1 + \Lambda^4)) h^* u' \Lambda^2 \\ &= 2q^{-2} \Lambda^6 + 13\Lambda^8 - \frac{3}{2} q^{-2} \Lambda^{10} - 14\Lambda^{12} + \frac{1}{2} q^{-2} \Lambda^{14} + 5\Lambda^{16} + O(q). \end{aligned}$$

Using again that $\tilde{\theta}_4(h) = 1 + q^2 \Lambda^2 + O(q^3)$, this gives

$$\begin{aligned} &-\text{Coeff}_{q^0} \left[\frac{1}{2} \tanh(h) (\tilde{\theta}_4(h)^{8(n+2)} - \tilde{\theta}_4(h)^{8(n+1)}) u' h^* \Lambda^2 \right] \\ &= (16n + 29)\Lambda^8 - (12n + 26)\Lambda^{12} + (4n + 9)\Lambda^{16}. \end{aligned}$$

Again one checks that this gives $(1 - \Lambda^4)^3 l_n - (1 + \Lambda^4) l_{n-1} = 0$. This shows (1).

(2) By Proposition 5.3, we have

$$\chi_0^{X, F+}(nF + 2G) = -\text{Coeff}_{q^0} \left[\frac{1}{2} \coth(2h) \tilde{\theta}_4(h)^{8(n+2)} u' h^* \Lambda^2 \right],$$

Using $-\frac{1}{2} \coth(2h) = \frac{1}{4} (-\coth(h) - \tanh(h))$ we get from (5.18) and (5.19) that

$$\chi_0^{X, F+}(nF + 2G) = \frac{1}{2} \left(\frac{1}{(1 - \Lambda^4)^{2n+3}} + \frac{(1 + \Lambda^4)^n}{(1 - \Lambda^4)^{3n+3}} \right) - 1 - (4n + 9)\Lambda^4,$$

and the claim follows. \square

Proof of Theorem 1.2. This follows directly from Proposition 5.14, Proposition 5.16, Proposition 5.17 and Proposition 3.22 and Definition 2.5. \square

5.4. Blowdown and the K -theoretic Donaldson invariants of \mathbb{P}^2 . Now we will prove Theorem 1.3, by combining the formulas Theorem 1.2 for $\widehat{\mathbb{P}}^2$ with the blowup formulas Lemma 2.3 and Lemma 4.33.

Proof of Theorem 1.3. (1) By Theorem 1.2 we have

$$\chi_0^{\widehat{\mathbb{P}}^2, H_+}(H) = \chi_0^{\widehat{\mathbb{P}}^2, H_+}(\frac{1}{2}F + G) = \frac{1}{(1 - \Lambda^4)^3} - 1 - \frac{17}{2}\Lambda^4.$$

We have $\chi_0^{\widehat{\mathbb{P}}^2, H}(H) = \chi_0^{\widehat{\mathbb{P}}^2, H_+}(H) - \frac{1}{2} \sum_{\xi} \delta_{\xi}^{\widehat{\mathbb{P}}^2}(H)$, where ξ runs over all classes of class 0 with $0 = \langle \xi, H \rangle > \langle \xi, E \rangle$. These are all the $\xi = 2nE$ with $n \in \mathbb{Z}_{\geq 0}$. By Theorem 3.19 we have that $\delta_{2nE}^{\widehat{\mathbb{P}}^2}(H) = 0$ unless $4n^2 \leq |2n| + 2$. This is only possible for $n = 1$ and direct computation gives $\delta_{2E}^{\widehat{\mathbb{P}}^2}(H) = \Lambda^4$. Thus

$$\chi_0^{\widehat{\mathbb{P}}^2, H}(H) = \frac{1}{(1 - \Lambda^4)^3} - 1 - 9\Lambda^4.$$

On the other hand we have by Lemma 2.3 $\chi_0^{\mathbb{P}^2, H}(H) = \chi_0^{\widehat{\mathbb{P}}^2, H_+}(H)$.

Alternatively observe that by Theorem 1.2

$$\chi_0^{\widehat{\mathbb{P}}^2, H_+}(H - E) = \chi_0^{\widehat{\mathbb{P}}^2, H_+}(F) = \frac{1}{(1 - \Lambda^4)^2} - 1 - 7\Lambda^4,$$

As above

$$\chi_0^{\widehat{\mathbb{P}}^2, H}(H - E) = \chi_0^{\widehat{\mathbb{P}}^2, H_+}(H - E) - \frac{1}{2} \sum_{n > 0} \delta_{2nE}^{\widehat{\mathbb{P}}^2}(H - E).$$

By Theorem 3.19 we have that $\delta_{2nE}^{\widehat{\mathbb{P}}^2}(H - E) = 0$ unless $4n^2 \leq |4n| + 2$. This is only possible for $n = 1$ and direct computation gives $\delta_{2E}^{\widehat{\mathbb{P}}^2}(H - E) = 2\Lambda^4 - 18\Lambda^8$. Thus we get by Lemma 4.33

$$(5.20) \quad \chi_0^{\mathbb{P}^2, H}(H) = \frac{\chi_0^{\widehat{\mathbb{P}}^2, H}(H - E) - \Lambda^4 + 9\Lambda^8}{(1 - \Lambda^4)} = \frac{1}{(1 - \Lambda^4)^3} - 1 - 9\Lambda^4.$$

The result follows because $\text{Coeff}_{\Lambda^d} [\chi_0^{\mathbb{P}^2, H}(H)] = \chi(M_H^{\mathbb{P}^2}(0, d), \mu(H))$ for $d > 4$.

(2) By Theorem 1.2 we have

$$\chi_0^{\widehat{\mathbb{P}}^2, H_+}(2H) = \chi_0^{\widehat{\mathbb{P}}^2, H_+}(F + 2G) = \frac{1}{(1 - \Lambda^4)^6} - 1 - 13\Lambda^4,$$

As above, using again Theorem 3.19

$$\chi_0^{\widehat{\mathbb{P}}^2, H}(2H) = \chi_0^{\widehat{\mathbb{P}}^2, H_+}(2H) - \frac{1}{2} \sum_{n > 0} \delta_{2nE}^{\widehat{\mathbb{P}}^2}(2H) = \chi_0^{\widehat{\mathbb{P}}^2, H_+}(2H) - \frac{1}{2} \delta_{2E}^{\widehat{\mathbb{P}}^2}(2H),$$

and $\delta_{2E}^{\widehat{\mathbb{P}}^2}(2H) = \Lambda^4$. Thus Lemma 2.3 gives

$$(5.21) \quad \chi_0^{\mathbb{P}^2, H}(2H) = \frac{1}{(1 - \Lambda^4)^6} - 1 - \frac{27}{2}\Lambda^4.$$

An alternative proof is by observing that

$$\chi_0^{\widehat{\mathbb{P}}^2, H_+}(2H - E) = \chi_0^{\widehat{\mathbb{P}}^2, H_+}(\frac{3}{2}F + G) = \frac{1}{(1 - \Lambda^4)^5} - 1 - \frac{23}{2}\Lambda^4,$$

Using Theorem 3.19 this gives

$$\chi_0^{\widehat{\mathbb{P}}^2, H}(2H - E) = \chi_0^{\widehat{\mathbb{P}}^2, H_+}(2H - E) - \frac{1}{2} \sum_{n>0} \delta_{2nE}^{\widehat{\mathbb{P}}^2}(2H - E) = \chi_0^{\widehat{\mathbb{P}}^2, H_+}(2H - E) - \frac{1}{2} \delta_{2E}^{\widehat{\mathbb{P}}^2}(2H - E),$$

and $\delta_{2E}^{\widehat{\mathbb{P}}^2}(2H - E) = 2\Lambda^4 - 27\Lambda^8$. Thus Lemma 4.33 gives again $\chi_0^{\mathbb{P}^2, H}(2H) = \frac{1}{(1-\Lambda^4)^6} - 1 - \frac{27}{2}\Lambda^4$.

(3) By Theorem 1.2 and Lemma 2.3 we have

$$(5.22) \quad \Lambda \chi_H^{\mathbb{P}^2, H}(2H) = \chi_F^{\widehat{\mathbb{P}}^2, H_+}(2H) = \chi_F^{\widehat{\mathbb{P}}^2, H_+}(F + 2G) = \frac{\Lambda^4}{(1 - \Lambda^4)^6},$$

The result follows because $\text{Coeff}_{\Lambda^d} [\chi_H^{\mathbb{P}^2, H}(2H)] = \chi(M_H^{\mathbb{P}^2}(0, d), \mu(H))$ for $d > 0$.

(4) By Theorem 1.2 we have

$$\chi_0^{\widehat{\mathbb{P}}^2, H_+}(3H - E) = \chi_0^{\widehat{\mathbb{P}}^2, H_+}(2F + 2G) = \frac{1 + \Lambda^8}{(1 - \Lambda^4)^9} - 1 - 17\Lambda^4.$$

Using Theorem 3.19 this gives

$$\chi_0^{\widehat{\mathbb{P}}^2, H}(3H - E) = \chi_0^{\widehat{\mathbb{P}}^2, H_+}(3H - E) - \frac{1}{2} \sum_{n>0} \delta_{2nE}^{\widehat{\mathbb{P}}^2}(3H - E) = \chi_0^{\widehat{\mathbb{P}}^2, H_+}(3H - E) - \frac{1}{2} \delta_{2E}^{\widehat{\mathbb{P}}^2}(3H - E),$$

and $\delta_{2E}^{\widehat{\mathbb{P}}^2}(3H - E) = 2\Lambda^4 - 38\Lambda^8$, this gives by Lemma 4.33

$$(5.23) \quad \chi_0^{\mathbb{P}^2, H}(3H) = \frac{1 + \Lambda^8}{(1 - \Lambda^4)^{10}} - 1 - 19\Lambda^4.$$

□

5.5. Some further invariants of the blowup of the plane. In this subsection we apply the blowup formula Lemma 2.3 to the results of the previous section to obtain K -theoretic invariants with respect to first Chern class H or E .

Corollary 5.24. (1) For $P = aH + bF$ with $\frac{b}{a} < 2$ we have

$$\begin{aligned} \Lambda + \sum_{d>4} \chi(M_P^{\widehat{\mathbb{P}}^2}(E, d), \mu(H)) \Lambda^d &= \frac{\Lambda}{(1 - \Lambda^4)^3}, \\ \Lambda + \sum_{d>4} \chi(M_P^{\widehat{\mathbb{P}}^2}(E, d), \mu(2H)) \Lambda^d &= \frac{\Lambda}{(1 - \Lambda^4)^6}, \\ \Lambda + \sum_{d>4} \chi(M_P^{\widehat{\mathbb{P}}^2}(E, d), \mu(3H)) \Lambda^d &= \frac{\Lambda + \Lambda^9}{(1 - \Lambda^4)^{10}}. \end{aligned}$$

(2) For $P = aH + bF$ with $\frac{b}{a} < 1$ we have

$$\sum_{d>0} \chi(M_P^{\widehat{\mathbb{P}}^2}(H, d), \mu(2H)) \Lambda^d = \frac{\Lambda^3}{(1 - \Lambda^4)^6}.$$

Proof. (1) Let $\xi \in E + 2H^2(X, \mathbb{Z})$ with $\langle H, \xi \rangle > 0 > \langle F, \xi \rangle$ and with $\delta_{\xi}^{\widehat{\mathbb{P}}^2}(kH) \neq 0$ for some k with $1 \leq k \leq 3$. Then Theorem 3.19 gives that $\xi = 2nF - (2m-1)E$ with $n, m \in \mathbb{Z}_{>0}$ and $8nm - 4n + (2m-1)^2 \leq |(4+2k)n - 2m+1| + 2$. Thus either $2m+1-2kn \geq 8mn + (2m-1)^2$, which is impossible, or $2kn-2m+3 \geq 8nm-8n+(2m-1)^2$, which implies $m = 1$. Thus if $P = aH + bF$ with $\frac{b}{a} < 2$ then there is no class ξ of type (E) with $\langle H, \xi \rangle > 0 > \langle P, \xi \rangle$ with $\delta_{\xi}^{\widehat{\mathbb{P}}^2}(kH) \neq 0$. Therefore

$$\chi_E^{\widehat{\mathbb{P}}^2, H_+}(kH) = \sum_{d>4} \chi(M_P^{\widehat{\mathbb{P}}^2}(E, d), \mu(kH)) \Lambda^d.$$

(a) (5.20) and Lemma 2.3 give

$$\Lambda \chi_0^{\mathbb{P}^2, H}(H) = \frac{\Lambda}{(1-\Lambda^4)^3} - \Lambda - 9\Lambda^5$$

By (4.11) we have thus

$$\chi_E^{\widehat{\mathbb{P}}^2, H_+}(H) = \Lambda \chi_0^{\mathbb{P}^2, H}(H) + (\langle -H, K_{\mathbb{P}^2} \rangle + \frac{K_{\mathbb{P}^2}^2 + H^2}{2} + 1) \Lambda^5 = \frac{\Lambda}{(1-\Lambda^4)^3} - \Lambda.$$

(b) (5.21) and Lemma 2.3 give

$$\Lambda \chi_0^{\mathbb{P}^2, H}(2H) = \frac{\Lambda}{(1-\Lambda^4)^6} - \Lambda - \frac{27}{2} \Lambda^5.$$

By (4.11) we have

$$\chi_E^{\widehat{\mathbb{P}}^2, H_+}(2H) = \Lambda \chi_0^{\mathbb{P}^2, H}(H) + \frac{27}{2} \Lambda^5 = \frac{\Lambda}{(1-\Lambda^4)^6} - \Lambda.$$

(c) (5.21), Lemma 2.3 and (4.11) give that

$$\chi_E^{\widehat{\mathbb{P}}^2, H_+}(3H) = \Lambda \left(\frac{1 + \Lambda^8}{1 - \Lambda^{10}} - 1 - 19\Lambda^4 \right) + 19\Lambda^5.$$

(2) Let $\xi \in H + 2H^2(X, \mathbb{Z})$ with $\langle H, \xi \rangle > 0 > \langle F, \xi \rangle$ and with $\delta_{\xi}^{\widehat{\mathbb{P}}^2}(2H) \neq 0$ for some d . Then Theorem 3.19 gives that $\xi = (2n-1)F - (2m-1)E$ with $n, m \in \mathbb{Z}_{>0}$ and $(4n-2)(2m-1) + (2m-1)^2 \leq |(8n-4-2m+1)| + 2$. Thus either $(4n-3)(2m-1) + (2m-1)^2 \leq -8n+6$, which is impossible, or $(4n-1)(2m-1) + (2m-1)^2 \leq 8n-2$, which implies $m = 1$.

Thus if $P = aH + bF$ with $\frac{b}{a} < 1$ then there is no class ξ of type (E) with $\langle H, \xi \rangle > 0 > \langle P, \xi \rangle$ with $\delta_{\xi}^{\widehat{\mathbb{P}}^2}(2H) \neq 0$. Therefore

$$\chi_H^{\widehat{\mathbb{P}}^2, H_+}(2H) = \sum_{d>0} \chi(M_P^{\widehat{\mathbb{P}}^2}(H, d), \mu(H)) \Lambda^d.$$

By (5.22) and Lemma 2.3 we have $\chi_H^{\widehat{\mathbb{P}}^2, H}(2H) = \chi_H^{\mathbb{P}^2, H}(2H) = \frac{\Lambda^3}{(1-\Lambda^4)^6}$. As H does not lie on a wall of type (H) , we get $\chi_H^{\widehat{\mathbb{P}}^2, H_+}(2H) = \chi_H^{\widehat{\mathbb{P}}^2, H}(2H)$. \square

6. LE POTIER'S STRANGE DUALITY.

We have briefly recalled the setting of strange duality conjecture in §2.5. We use the same notations as in §2.5, Theorem 1.2 and Theorem 1.3. We will prove Theorem 1.4 in this section.

Remark 6.1. (1) If $c^* = (0, L, \chi = 0)$, then $M_H^X(c^*)$ does not depend on the polarization H , hence any ample class H is c^* -general. Actually if c^* is of rank 0, then the morphism $\lambda : K_{c^*, H} \rightarrow \text{Pic}(M_H^X(c^*))$ introduced in §2.1 can be extended to K_{c^*} for any ample H .

(2) H may not be c -general with $c = (2, 0, c_2)$, then the determinant line bundle $\mathcal{D}_{c, c^*} = \lambda(c^*)$ is not well-defined over the whole space $M_H^X(c)$, unless $\langle L, \xi \rangle = 0$ for every ξ of type c with $\langle H, \xi \rangle = 0$ and $\xi^2 + 4c_2 \in 8\mathbb{Z}_{\geq 0}$. One can see this from the construction of $\lambda(c^*)$: we first have a good $GL(V)$ -quotient $\Omega(c) \rightarrow M_H^X(c)$ with $\Omega(c)$ an open subset of some Quot-scheme. There is a universal family \mathcal{F} over $\Omega(c)$, and we get $\lambda(c^*)$ by descending the determinant line bundle $\lambda_{\mathcal{F}}(c^*)$ over $\Omega(c)$ to $M_H^X(c)$. $\lambda_{\mathcal{F}}(c^*)$ is $GL(V)$ -linearized and $\lambda(c^*)$ is well-defined if and only if $\lambda_{\mathcal{F}}(c^*)$ satisfies the descent condition (see Theorem 4.2.15 in [13]). Hence we know that $\lambda(c^*)$ is certainly well-defined over the stable locus $M_H^X(c)^s$ since $c^* \in K_c$. Denote this line bundle by \mathcal{D}_{c, c^*}^s . There are also strictly semistable points in $M_H^X(c)$, which correspond to S-equivalence classes of sheaves $\mathcal{I}_Z(\xi) \oplus \mathcal{I}_W(-\xi)$, with $\langle H, \xi \rangle = 0$ and $\xi^2 + 4c_2 \in 8\mathbb{Z}_{\geq 0}$ and $\text{len}(Z) = \text{len}(W) = \xi^2/8 + c_2/2$. Let $c_{+\xi}$ (resp. $c_{-\xi}$) be the class of $\mathcal{I}_Z(\xi)$ (resp. $\mathcal{I}_W(-\xi)$) in $K(X)$. Then by the descent condition, $\lambda(c^*)$ is well-defined over the strictly semistable point $[\mathcal{I}_Z(\xi) \oplus \mathcal{I}_W(-\xi)]$ if and only if $\chi(c^* \otimes c_{\pm\xi}) = 0$ (see the proof of Theorem 8.1.5 in [13]) which is equivalent to say that $\langle L, \xi \rangle = 0$ since $c^* = (0, L, \chi = 0)$.

Denote by $M_H^X(c)^g$ the biggest open subset of $M_H^X(c)$ where $\lambda(c^*)$ is well-defined. We denote this line bundle over $M_H^X(c)^g$ by \mathcal{D}_{c, c^*}^g . Notice that $M_H^X(c)^s \subset M_H^X(c)^g$ and by Remark 6.1 sheaves of the form $\mathcal{I}_Z(c_1/2) \oplus \mathcal{I}_W(c_1/2)$ are in $M_H^X(c)^g$, where Z, W are 0-dimensional subschemes of length $d/8$. This is because in this case $\xi = 0$. We have $M_H^X(c)^g = M_H^X(c)$ if the polarization H is c -general or proportional to L . Replacing $M_H^X(c)$ by $M_H^X(c)^g$ and \mathcal{D}_{c, c^*} by \mathcal{D}_{c, c^*}^g , we can define a morphism

$$(6.2) \quad SD_{c, c^*}^g : H^0(M_H^X(c)^g, \mathcal{D}_{c, c^*}^g)^\vee \rightarrow H^0(M_H^X(c^*), \mathcal{D}_{c^*, c}).$$

analogously to (2.10). We will prove the following theorem which implies Theorem 1.4.

Theorem 6.3. *SD_{c, c^*}^g is an isomorphism for the three cases of Theorem 1.4.*

6.1. Numerical condition. We first show that the two vector spaces $H^0(M_H^X(c), \mathcal{D}_{c, c^*})$ (or $H^0(M_H^X(c)^g, \mathcal{D}_{c, c^*}^g)$ in general) and $H^0(M_H^X(c^*), \mathcal{D}_{c^*, c})$ have the same dimension.

Proposition 6.4. *For the three cases of Theorem 1.4, we have*

$$(6.5) \quad \dim H^0(M_H^X(c)^g, \mathcal{D}_{c, c^*}^g) = \dim H^0(M_H^X(c)^s, \mathcal{D}_{c, c^*}^s) = \dim H^0(M_H^X(c^*), \mathcal{D}_{c^*, c}).$$

In particular if H is c -general or proportional to L , then $M_H^X(c)^g = M_H^X(c)$ and

$$(6.6) \quad \dim H^0(M_H^X(c), \mathcal{D}_{c, c^*}) = \dim H^0(M_H^X(c)^s, \mathcal{D}_{c, c^*}^s) = \dim H^0(M_H^X(c^*), \mathcal{D}_{c^*, c}).$$

Proof. Since $c^* = (0, L, \chi = -\langle \frac{c_1(L)}{2}, c_1 \rangle)$ for the three cases of Theorem 1.4, $\mathcal{D}_{c,c^*} = \mu(L)$.

In the cases (1) and (2) of Theorem 1.4, every strictly semistable sheaf is S-equivalent to a sheaf of the form $\mathcal{I}_Z(\xi/2) \oplus \mathcal{I}_W(-\xi/2)$ with ξ a class of type c such that $\langle H, \xi \rangle = 0$ and 0-dimensional subschemes Z, W of length $(4c_2 + \xi^2)/8 \leq c_2/2$. Hence the locus $M_H^X(c)^{sss}$ of strictly semistable sheaves is of codimension ≥ 2 in $M_H^X(c)$ because $\dim M_H^X(c)^s = 4c_2 - 3$ and $c_2 > 2$.

Furthermore $-K_X$ is ample and hence $M_H^X(c)$ has only rational singularities. Therefore we have

$$(6.7) \quad \dim H^0(M_H^X(c)^g, \mathcal{D}_{c,c^*}^g) = \dim H^0(M_H^X(c)^s, \mathcal{D}_{c,c^*}^s).$$

The dimension of $H^0(M_H^X(c^*), \mathcal{D}_{c^*,c})$ has been computed in [31] (see Theorem 4.4.1 and Theorem 4.5.2 in [31]) for cases (1) and (2). Theorem 1.2, Theorem 1.3 and Proposition 2.9 provide the dimension of $H^0(M_H^X(c), \mathcal{D}_{c,c^*})$ for H c -general. By comparing those two results, we get (6.6) for cases (1) and (2) with c -general polarization.

Now we assume that the polarization H lies on a wall W^ξ with ξ a class of type c . We want to show (6.5) for cases (1) and (2). With no loss of generality, we assume $\langle \xi, K_X \rangle \leq 0$. Let H_+ be a c -general polarization lying in the chamber next to W^ξ such that $\langle \xi, H_+ \rangle > 0$. We have a surjective birational map

$$\rho : M_{H_+}^X(c)^s \dashrightarrow M_H^X(c)^s,$$

by sending every sheaf to itself, which is an isomorphism outside the locus E_{H_+} consisting of H_+ -stable sheaves lying in the following exact sequence

$$(6.8) \quad 0 \rightarrow \mathcal{I}_Z(-\tilde{\xi}/2) \rightarrow \mathcal{F} \rightarrow \mathcal{I}_W(\tilde{\xi}/2) \rightarrow 0,$$

with $\langle \tilde{\xi}, H_+ \rangle > 0$ and $\langle \tilde{\xi}, H \rangle = 0$, and Z, W 0-dimensional subschemes satisfying $\text{len}(Z) + \text{len}(W) = c_2 + \tilde{\xi}^2/4$. Because now we have $\text{Pic}(X) \cong H^2(X, \mathbb{Z})$ is free of rank 2, we must have $\tilde{\xi} = a\xi$ for some $a > 0$ and hence $\langle \tilde{\xi}, K_X \rangle \leq 0$. It is obvious that ρ identifies the determinant line bundles \mathcal{D}_{c,c^*} on both sides. Hence to show (6.5) for H , it is enough to show E_{H_+} is of codimension ≥ 2 in $M_{H_+}^X(c)^s$, which has pure dimension $d - 3 = 4c_2 - 3$.

$\text{Hom}(\mathcal{I}_W(\tilde{\xi}/2), \mathcal{I}_Z(-\tilde{\xi}/2)) = 0$ because $\langle H_+, \tilde{\xi} \rangle > 0$. Since $-K_X$ is ample and $\langle H, \tilde{\xi} \rangle = 0$, we have $\text{Ext}^2(\mathcal{I}_W(\tilde{\xi}/2), \mathcal{I}_Z(-\tilde{\xi}/2)) \cong \text{Hom}(\mathcal{I}_Z(-\tilde{\xi}), \mathcal{I}_W(\tilde{\xi} + K_X))^\vee = 0$. Hence

$$\dim \text{Ext}^1(\mathcal{I}_W(\tilde{\xi}/2), \mathcal{I}_Z(-\tilde{\xi}/2)) = -\chi(\mathcal{I}_W(\tilde{\xi}/2), \mathcal{I}_Z(-\tilde{\xi}/2)) = \langle \tilde{\xi} \cdot K_X \rangle / 2 - \tilde{\xi}^2/4 + c_2.$$

Hence

$$\dim E_{H_+} \leq \max_{\substack{\tilde{\xi}/2 \in H^2(X, \mathbb{Z}), 2\tilde{\xi} \in \mathbb{Z}_{\geq 0} \\ \langle \tilde{\xi}, H_+ \rangle > 0, \langle \tilde{\xi}, H \rangle = 0}} \{3c_2 + \tilde{\xi}^2/4 + \langle \tilde{\xi}, K_X \rangle / 2 - 1\}.$$

Since $\langle \xi, K_X \rangle \leq 0$, $\dim E_{H_+} \leq \max_{\substack{\tilde{\xi}/2 \in H^2(X, \mathbb{Z}), 2l_{\tilde{\xi}} \in \mathbb{Z}_{\geq 0} \\ \langle \tilde{\xi}, H_+ \rangle > 0, \langle \tilde{\xi}, H \rangle = 0}} \{3c_2 + \tilde{\xi}^2/4 - 1\}$. Then

$$(6.9) \quad N_c := \dim M_{H_+}(c)^s - \dim E_{H_+} \geq \min_{\substack{\tilde{\xi}/2 \in H^2(X, \mathbb{Z}), 2l_{\tilde{\xi}} \in \mathbb{Z}_{\geq 0} \\ \langle \tilde{\xi}, H_+ \rangle > 0, \langle \tilde{\xi}, H \rangle = 0}} \{c_2 - 2 - \tilde{\xi}^2/4\}.$$

Since $\tilde{\xi}/2 \in H^2(X, \mathbb{Z})$ and $\tilde{\xi}^2 < 0$, $-\tilde{\xi}^2/4 \geq 1$. Moreover since $c_2 > 2$, we have $c_2 - 2 - \tilde{\xi}^2/4 \geq 2$ and hence $N_c \geq 2$. This proves the claim for cases (1) and (2).

For case (3), $M_H^X(c) = M_H^X(c)^s = M_H^X(c)^g$ and there is no wall. By Theorem 3.5 in [15] $M_H^X(c^*) \cong |2H| \cong \mathbb{P}^5$ and moreover a sheaf $\mathcal{G} \in M_H^X(c^*)$ with support $C_{\mathcal{G}}$ is isomorphic to $\mathcal{O}_{C_{\mathcal{G}}} \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$. We have $c = (2, H, c_2)$. If $c_2 = 1$, then $M_H^X(c)$ consists of only one point $[\mathcal{T}_{\mathbb{P}^2}(-1)]$ with $\mathcal{T}_{\mathbb{P}^2}$ the tangent bundle over \mathbb{P}^2 . $H^0(\mathcal{T}_{\mathbb{P}^2}(-2) \otimes \mathcal{O}_C) = 0$ for any $C \in |2H|$. Hence $\mathcal{D}_{c^*,c} \cong \mathcal{O}_{|2H|}$ for $c_2 = 1$ and for $c_2 > 1$ we have $\mathcal{D}_{c^*,c} \cong \mathcal{O}_{|2H|}(c_2 - 1)$ by the following lemma due to Le Potier.

Lemma 6.10 (Proposition 2.8 in [16]). *If x is not a base point of $|L|$, then the determinant line bundle $\lambda_x := \lambda([\mathcal{O}_x])$ associated to the skyscraper sheaf $[\mathcal{O}_x]$ over $M(0, L, \chi)$ satisfies $\lambda_x \cong \pi^* \mathcal{O}_{|L|}(-1)$, where $\pi : M(0, L, \chi) \rightarrow |L|$ is the projection sending every sheaf to its support.*

By Theorem 1.3 and Proposition 2.9 we have for case (3)

$$\dim H^0(M(c), \mathcal{D}_{c,c^*}) = \dim H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(c_2 - 1)) = \dim H^0(M(c^*), \mathcal{D}_{c^*,c}) = \binom{c_2 + 4}{c_2 - 1}.$$

This finishes the proof of the proposition. \square

6.2. Strange duality. Recall that we have introduced in Section 2.5 the locus

$$\mathcal{D} := \{([\mathcal{F}], [\mathcal{G}]) \in M_H^X(c) \times M_H^X(c^*) \mid H^0(X, \mathcal{F} \otimes \mathcal{G}) \neq 0\},$$

which gives a canonical section σ_{c,c^*} of the line bundle $\mathcal{D} := \mathcal{D}_{c,c^*} \boxtimes \mathcal{D}_{c^*,c} \in \text{Pic}(M_H^X(c) \times M_H^X(c^*))$ and induces the strange duality map

$$(6.11) \quad SD_{c,c^*} : H^0(M_H^X(c), \mathcal{D}_{c,c^*})^\vee \rightarrow H^0(M_H^X(c^*), \mathcal{D}_{c^*,c}).$$

If the polarization is not c -general, we have the map

$$(6.12) \quad SD_{c,c^*}^g : H^0(M_H^X(c)^g, \mathcal{D}_{c,c^*}^g)^\vee \rightarrow H^0(M_H^X(c^*), \mathcal{D}_{c^*,c}).$$

To show Theorem 1.4 (resp. Theorem 6.3), it is by Proposition 6.4 enough to show that SD_{c,c^*} (resp. SD_{c,c^*}^g) is surjective.

Let \mathcal{F} be a semistable sheaf of class c . Denote by $s_{\mathcal{F}}$ the section of $\mathcal{D}_{c^*,c}$ over $M_H^X(c^*)$ up to scalars defined by the canonical divisor $D_{\mathcal{F}} := \{\mathcal{G} \in M_H^X(c^*) \mid H^0(\mathcal{F} \otimes \mathcal{G}) \neq 0\}$.

Lemma 6.13. *$s_{\mathcal{F}}$ only depends on the S -equivalence class of \mathcal{F} . More precisely, take a Jordan-Hölder filtration of \mathcal{F}*

$$0 = JH_0 \subset JH_1 \subset \dots \subset JH_\ell = \mathcal{F}$$

and let $J_i := JH_i/JH_{i-1}$, then $s_{\mathcal{F}} = \Pi_{i \geq 1} s_{J_i}$.

Proof. It is enough to show that $D_{\mathcal{F}} = \cup_{i \geq 1} D_{J_i}$. Since the strictly semistable locus in $M_H^X(c^*)$ forms a subset of codimension ≥ 2 (See Proposition 3.4 in [15] and Lemma 4.2.6 in [31]), It is enough to show that $D_{\mathcal{F}}$ and $\cup_{i \geq 1} D_{J_i}$ coincide on the stable locus $M_H^X(c^*)^s$. We write down an exact sequence

$$(6.14) \quad 0 \rightarrow J_1 \rightarrow \mathcal{F} \rightarrow \mathcal{F}/J_1 \rightarrow 0.$$

Take any \mathcal{G} stable sheaf of class c^* . We tensor (6.14) by \mathcal{G} and get

$$(6.15) \quad 0 \rightarrow \mathcal{G} \otimes J_1 \rightarrow \mathcal{G} \otimes \mathcal{F} \rightarrow \mathcal{G} \otimes \mathcal{F}/J_1 \rightarrow 0,$$

and

$$(6.16) \quad 0 \rightarrow H^0(\mathcal{G} \otimes J_1) \rightarrow H^0(\mathcal{G} \otimes \mathcal{F}) \rightarrow H^0(\mathcal{G} \otimes \mathcal{F}/J_1) \rightarrow H^1(\mathcal{G} \otimes J_1).$$

Notice that $\chi(\mathcal{G} \otimes J_1) = h^0(\mathcal{G} \otimes J_1) - h^1(\mathcal{G} \otimes J_1) = 0$. By (6.16) we see that $H^0(\mathcal{G} \otimes \mathcal{F}) \neq 0 \Leftrightarrow H^0(\mathcal{G} \otimes J_1) \neq 0$ or $H^0(\mathcal{G} \otimes \mathcal{F}/J_1) \neq 0$. Hence we get $D_{\mathcal{F}} = D_{J_1} \cup D_{\mathcal{F}/J_1}$, and the lemma follows by applying the induction assumption to \mathcal{F}/J_1 . \square

We then have the following criterion for the surjectivity of maps SD_{c,c^*} and SD_{c,c^*}^s .

Proposition 6.17. *If there are finitely many semistable sheaves \mathcal{F}_i $i \in I$ of class c such that $\{s_{\mathcal{F}_i}\}_{i \in I}$ spans $H^0(M_H^X(c^*), \mathcal{D}_{c^*,c})$, then the map SD_{c,c^*} in (6.11) is surjective. Moreover, if \mathcal{F}_i can be chosen to be in $M_H^X(c)^g$, then SD_{c,c^*}^g in (6.12) is surjective.*

Proof. This is very standard in linear algebra. We have the section σ_{c,c^*} associated to the divisor \mathcal{D} over $M_H^X(c) \times M_H^X(c^*)$. Let $\{e_k\}_{1 \leq k \leq m}$ and $\{f_j\}_{1 \leq j \leq l}$ be the basis of $H^0(M_H^X(c^*), \mathcal{D}_{c^*,c})$ and $H^0(M_H^X(c), \mathcal{D}_{c,c^*})$ respectively. Then we have $\sigma_{c,c^*} = \sum a_{kj} e_k \otimes f_j$ with a_{kj} constant. Thus we have a $m \times l$ matrix $A = (a_{kj})$, and the map SD_{c,c^*} is surjective if and only if A is of rank m .

On the other hand $s_{\mathcal{F}_i} = \sum a_{kj} \cdot f_j([\mathcal{F}_i]) \cdot e_k$, where $f_j([\mathcal{F}_i])$ is the value of f_j at point $[\mathcal{F}_i]$. Because $\{s_{\mathcal{F}_i}\}$ spans $H^0(M_H^X(c^*), \mathcal{D}_{c^*,c})$, we can recover e_k as a linear combination of the $s_{\mathcal{F}_i}$, hence we can find an $l \times m$ matrix B such that $A \cdot B = Id_m$, hence A is of rank m . The statement for SD_{c,c^*}^g also follows if $[\mathcal{F}_i] \in M(c)^g$ for all i . \square

We will use Proposition 6.17 to show the surjectivity of SD_{c,c^*}^g for the three cases of Theorem 1.4.

† cases (1) and (2).

In these two cases we have $c^* = (0, L, \chi = 0)$ with $L = -K_X$ or $L = -K_X + F$ and $c = (2, 0, c_2)$.

If $c^* = (0, L, \chi = 0)$ with L some effective line bundle, we denote by π the projection $M_H^X(c^*) \rightarrow |L|$ sending every sheaf to its support. Denote by Θ the determinant line bundle associated to $[\mathcal{O}_X]$ over $M_H^X(c^*)$. Therefore by Lemma 6.10 we have $\mathcal{D}_{c^*, c} \cong \Theta^2(c_2) := \Theta^2 \otimes \pi^* \mathcal{O}_{|L|}(c_2)$ with $c = (2, 0, c_2)$. Notice that these notations are compatible with those in [31].

Lemma 6.18. *For $c^* = (0, L, \chi = 0)$ with $L = -K_X$ or $L = -K_X + F$, the multiplication map*

$$m_1 : H^0(M_H^X(c^*), \Theta) \otimes H^0(M_H^X(c^*), \pi^* \mathcal{O}_{|L|}(c_2)) \rightarrow H^0(M_H^X(c^*), \Theta(c_2)),$$

is surjective.

Proof. By Lemma 4.3.3 in [31], $\pi_* \mathcal{O}_{M_H^X(c^*)} \cong \mathcal{O}_{|L|}$. Thus $\pi_*(\pi^* \mathcal{O}_{|L|}(n)) \cong \mathcal{O}_{|L|}(n)$. By Theorem 4.4.1 and Theorem 4.5.2 in [31], $\pi_* \Theta \cong \mathcal{O}_{|L|}$ and the multiplication map

$$\tilde{m}_1 : H^0(|L|, \pi_* \Theta) \otimes H^0(|L|, \pi_* \pi^*(\mathcal{O}_{|L|}(n))) \rightarrow H^0(|L|, \pi_* \Theta(n))$$

is surjective. Hence so is m_1 . \square

We choose a finite collection of distinct points $\{x_j\}_{j \in J}$ on X , and associate to each point x_j a divisor in $|L|$ consisting of curves passing through x_j , which gives a section t_{x_j} of $\mathcal{O}_{|L|}(1)$. Let $d_L = \dim |L|$, then it is possible to choose $d_L + 1$ distinct points x_j such that $\{t_{x_j}\}_{j=1}^{d_L+1}$ spans $H^0(\mathcal{O}_{|L|}(1))$. Hence we can choose $n(d_L + 1)$ distinct points x_j^k with $1 \leq j \leq d_L + 1$, $1 \leq k \leq n$ such that $\{t_{j_1, \dots, j_n}\}$ spans $H^0(\mathcal{O}_{|L|}(n))$, where t_{j_1, \dots, j_n} is defined as follows.

$$t_{j_1, \dots, j_n} := \prod_{k=1}^n t_{x_{j_k}^k}, \text{ with } t_{x_{j_k}^k} \text{ the section associated to } x_{j_k}^k.$$

Let R be a subset of $\{x_j^k\}$ and denote by \mathcal{I}_R the ideal sheaf of points appearing in R . Then $s_{\mathcal{I}_R} = s_{\mathcal{O}_X} \times \pi^* t_R$ where $t_R := \prod_{x \in R} t_x$. For any two disjoint subsets R and T of $\{x_j^k\}$ such that $\#R = \lfloor \frac{n}{2} \rfloor$ is the round-down of $\frac{n}{2}$ and $\#T = n - \lfloor \frac{n}{2} \rfloor$, we define a rank 2 torsion free sheaf $\mathcal{F}_{R,T}$ to be an extension of \mathcal{I}_R by \mathcal{I}_T , i.e.

$$(6.19) \quad 0 \rightarrow \mathcal{I}_T \rightarrow \mathcal{F}_{R,T} \rightarrow \mathcal{I}_R \rightarrow 0.$$

Moreover if $n \geq 2$ and n is odd, we ask (6.19) not to split. It is easy to see that for all $n \geq 2$ or $n = 0$, $\mathcal{F}_{R,T}$ is semistable of class $c = (2, 0, c_2 = n)$ and $s_{\mathcal{F}_{R,T}} = s_{\mathcal{I}_T} \cdot s_{\mathcal{I}_R}$. Hence we have the following lemma as an easy corollary of Lemma 6.18

Lemma 6.20. *For $n \geq 2$ or $n = 0$, the set $\{s_{\mathcal{F}_{R,T}}\}_{R,T}$ spans the image of $H^0(M_H^X(c^*), \Theta(c_2))$ in $H^0(M_H^X(c^*), \Theta^2(c_2))$ via the natural embedding induced by the following sequence*

$$(6.21) \quad 0 \rightarrow \Theta(c_2) \rightarrow \Theta^2(c_2) \rightarrow \Theta^2(c_2)|_{D_\Theta} \rightarrow 0,$$

where $D_\Theta := \{\mathcal{G} \in M_H^X(c^*) \mid H^0(\mathcal{G}) \neq 0\}$ is the canonical divisor associated to Θ .

Remark 6.22. Notice that $[\mathcal{F}_{R,T}] \in M_H^X(c)^g$ under any polarization.

By Theorem 4.4.1 and Theorem 4.5.2 in [31], $\Theta(n)$ has no higher cohomology for $n \geq 0$. Then by (6.21) we have the exact sequence

$$(6.23) \quad 0 \rightarrow H^0(\Theta(n)) \rightarrow H^0(\Theta^2(n)) \xrightarrow{\varpi} H^0(D_\Theta, \Theta^2(n)|_{D_\Theta}) \rightarrow 0.$$

Lemma 6.24. *The multiplication map*

$$m_2 : H^0(D_\Theta, \Theta^2(2)|_{D_\Theta}) \otimes H^0(M_H^X(c^*), \pi^*(\mathcal{O}_{|L|}(n-2))) \rightarrow H^0(D_\Theta, \Theta^2(n)|_{D_\Theta})$$

is surjective.

Proof. By Lemma 4.4.4 and Lemma 4.5.4 in [31], the multiplication map

$$\tilde{m}_2 : H^0(|L|, \pi_*(\Theta^2(2)|_{D_\Theta})) \otimes H^0(|L|, \pi_*\pi^*(\mathcal{O}_{|L|}(n-2))) \rightarrow H^0(|L|, \pi_*(\Theta^2(n)|_{D_\Theta}))$$

is surjective. Hence so is m_2 since $\pi_*(\pi^*\mathcal{O}_{|L|}(n-2)) \cong \mathcal{O}_{|L|}(n-2)$. \square

Proposition 6.25. *We can find a set of finitely many slope-stable vector bundles $\{\mathcal{E}_2^i\}$ of class $(2, 0, c_2 = 2)$ such that the images of $s_{\mathcal{E}_2^i}$ in $H^0(D_\Theta, \Theta^2(2)|_{D_\Theta})$ via map ϖ in (6.23) span $H^0(D_\Theta, \Theta^2(2)|_{D_\Theta})$.*

Proof. For case (1), we have $D_\Theta \cong |L|$ and $\Theta^2(2) \cong \mathcal{O}_{|L|}$ by Lemma 4.4.3 and Lemma 4.4.4 in [31]. It is enough to show that there exists a slope-stable vector bundle \mathcal{E}_2 of class $(2, 0, c_2 = 2)$, such that for all $\mathcal{G} \in D_\Theta$, $H^0(\mathcal{G} \otimes \mathcal{E}_2) = 0$. If we are on \mathbb{P}^2 , then any semistable bundle of class $(2, 0, c_2 = 2)$ is slope-stable. If we are on the two Hirzebruch surfaces, then by Lemma 6.27 and Remark 6.28 below there are slope-stable bundles of class $(2, 0, c_2 = 2)$ for any polarization.

Notice that in this case $\mathcal{G} \in D_\Theta \Leftrightarrow \mathcal{G} \cong \mathcal{O}_{C_\mathcal{G}}$ with $C_\mathcal{G}$ the support of \mathcal{G} . Let \mathcal{E}_2 be any slope-stable vector bundle of class $(2, 0, c_2 = 2)$. We want to show that $H^0(\mathcal{O}_C \otimes \mathcal{E}_2) = 0$ for all curves $C \in |L|$. We have the following exact sequence

$$0 \rightarrow K_X \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_C \rightarrow 0.$$

Tensoring it by \mathcal{E}_2 and taking global sections, we get

$$0 \rightarrow H^0(K_X \otimes \mathcal{E}_2) \rightarrow H^0(\mathcal{E}_2) \rightarrow H^0(\mathcal{E}_2 \otimes \mathcal{O}_C) \rightarrow H^1(K_X \otimes \mathcal{E}_2).$$

By stability we have $H^0(\mathcal{E}_2) = H^2(\mathcal{E}_2) = 0$ and moreover since $\chi(\mathcal{E}_2) = 0$, we have $H^1(\mathcal{E}_2) = H^1(K_X \otimes \mathcal{E}_2)^\vee = 0$. Because \mathcal{E}_2 is a rank 2 bundle with trivial determinant, we have $\mathcal{E}_2^\vee \cong \mathcal{E}_2$. Hence $H^1(\mathcal{E}_2) = H^1(K_X \otimes \mathcal{E}_2) = 0$, hence $H^0(\mathcal{E}_2 \otimes \mathcal{O}_C) = 0$ for all $C \in |L|$. This finishes the proof for case (1).

Now we deal with Case (2). In this case $X = \mathbb{P}^1 \times \mathbb{P}^1$ or $\widehat{\mathbb{P}}^2$. We can write $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e))$ with $e = 0$ for $\mathbb{P}^1 \times \mathbb{P}^1$ and $e = 1$ for $\widehat{\mathbb{P}}^2$.

Define the section class $\Xi := G - eF/2$. Then $\Xi^2 = -e$ and $a\Xi + bF$ is ample if and only if $a, b > 0$ and $b > ae$. Choose three distinct points x_1, x_2, x_3 on X such that no two of them lie on a divisor of class F or Ξ . Denote by \mathcal{I}_j the ideal sheaf of $\{x_i, i \neq j\}$, hence we have $H^0(\mathcal{I}_j(F)) =$

$H^0(\mathcal{I}_j(\Xi)) = 0$ for all $1 \leq j \leq 3$. It is easy to compute that $\text{Ext}^1(\mathcal{I}_j(F), \mathcal{O}_X(-F)) \neq 0$. Let \mathcal{E}_2^i be a vector bundle lying in the following exact sequence.

$$(6.26) \quad 0 \rightarrow \mathcal{O}_X(-F) \rightarrow \mathcal{E}_2^i \rightarrow \mathcal{I}_j(F) \rightarrow 0.$$

There exists such vector bundle is because the Cayley Bacharach condition is fulfilled by $H^0(K_X(2F)) = 0$. Lemma 6.27, Remark 6.28 and Lemma 6.29 below imply the statement for Case (2). This proves the proposition. \square

To deal with Case (2), we have the following three lemmas.

Lemma 6.27. *If $e = 1$, \mathcal{E}_2^i is slope-stable for any polarization. If $e = 0$, \mathcal{E}_2^i is slope-stable for the polarization $G + aF$ for $a \geq 1$.*

Proof. Choose any polarization $P = \Xi + \nu F$ with $\nu \in \mathbb{Q}, \nu > e$. Since \mathcal{E}_2^i is locally free, it is enough to show that for any divisor $S = a\Xi + bF$ with $a, b \in \mathbb{Z}$ such that $\langle S, P \rangle \geq 0$, we have $\text{Hom}(\mathcal{O}_X(S), \mathcal{E}_2^i) = 0$. By (6.26) it is enough to show $\text{Hom}(\mathcal{O}_X(S), \mathcal{O}_X(-F)) = 0 = H^0(\mathcal{O}_X(S), \mathcal{I}_j(F))$.

We have $\text{Hom}(\mathcal{O}_X(S), \mathcal{O}_X(-F)) = H^0(\mathcal{O}_X(-a\Xi - (b+1)F)) = 0$ because $\langle P, (-a\Xi - (b+1)F) \rangle < 0$. If $\text{Hom}(\mathcal{O}_X(S), \mathcal{I}_j(F)) = H^0(\mathcal{I}_j(-a\Xi - (b-1)F)) \neq 0$, then $a \leq 0$, $b \leq 1$ and $-a\Xi - (b-1)F \neq 0$. But $\langle (a\Xi + bF), P \rangle = (\nu - e)a + b \geq 0$, hence we have $0 > a \geq -1/(\nu - e)$ and $b = 1$ or $a = b = 0$.

If $e = 1$, then $H^0(\mathcal{O}_X(k\Xi)) = H^0(\mathcal{O}_X(\Xi)) \cong \mathbb{C}$ for all $k \geq 1$. Hence $H^0(\mathcal{I}_j(k\Xi)) = H^0(\mathcal{I}_j(F)) = 0$ for any k because \mathcal{I}_j is an ideal sheaf of two distinct points not lying on a divisor of class F or Ξ . Hence $\text{Hom}(\mathcal{O}_X(S), \mathcal{I}_j(F)) = 0$ and hence \mathcal{E}_2^i is slope-stable for any polarization.

If $e = 0$, then we ask $\nu \geq 1$ and hence $a = -1$, $b = 1$ or $a = b = 0$. But $H^0(\mathcal{I}_j(\Xi)) = H^0(\mathcal{I}_j(F)) = 0$ and hence \mathcal{E}_2^i is slope-stable. \square

Remark 6.28. In Lemma 6.27, if $X = \mathbb{P}^1 \times \mathbb{P}^1$ then $G = \Xi$ and F are symmetric, hence we can always write a polarization as $G + aF$ with $a \geq 1$. Hence for any polarization, we can construct slope-stable vector bundles \mathcal{E}_2^i .

Lemma 6.29. *The restrictions of $s_{\mathcal{E}_2^i}, 1 \leq i \leq 3$ to D_Θ span $H^0(D_\Theta, \Theta^2(2)|_{D_\Theta})$.*

Proof. It is enough to choose three distinct points $\mathcal{G}^1, \mathcal{G}^2, \mathcal{G}^3 \in D_\Theta$ such that $s_{\mathcal{E}_2^i}(\mathcal{G}^j) \neq 0$ if and only if $i = j$. In other words, $H^0(\mathcal{G}^j \otimes \mathcal{E}_2^i) = 0$ if and only if $i = j$.

Recall that we have chosen three distinct points x_1, x_2, x_3 . Denote by f_i the fiber passing through x_i , then $f_i \cong \mathbb{P}^1$ and $f_i \cap f_j = \emptyset$ for $i \neq j$. Choose a smooth curve $C \in |-K_X|$. Then we define \mathcal{G}^i to be (S -equivalent to) $\mathcal{O}_C \oplus \mathcal{O}_{f_i}(-1)$, where $\mathcal{O}_{f_i}(-1) \cong \mathcal{O}_{\mathbb{P}^1}(-1)$.

By Lemma 6.25 we see that $H^0(\mathcal{E}_2^i \otimes \mathcal{O}_C) = 0$ for $1 \leq i \leq 3$. Therefore $H^0(\mathcal{E}_2^i \otimes \mathcal{G}^j) = 0 \Leftrightarrow H^0(\mathcal{E}_2^i \otimes \mathcal{O}_{f_j}(-1)) = 0$. By (6.26) we see that $\mathcal{E}_2^i|_{f_i} \cong \mathcal{O}_{f_i}^{\oplus 2}$, while for $i \neq j$, $\mathcal{E}_2^i|_{f_j} \cong \mathcal{O}_{f_i}(-1) \oplus \mathcal{O}_{f_i}(1)$. Hence $H^0(\mathcal{G}^j \otimes \mathcal{E}_2^i) = 0$ if and only if $i = j$. This proves the lemma. \square

Let W be a subset of $\{x_j^k\}_{1 \leq j \leq d_L+1}^{1 \leq k \leq n-2}$ such that $\#W = n - 2$. Let $\mathcal{O}_W \cong \bigoplus_{x \in W} \mathcal{O}_x$ be the structure sheaf of the subscheme W . Choose a surjective map $h^i : \mathcal{E}_2^i \rightarrow \mathcal{O}_W$ (one may choose h^i to factor through $\mathcal{E}_2^i \rightarrow \mathcal{E}_2^i \otimes \mathcal{O}_W \cong \mathcal{O}_W^{\oplus 2}$). Let \mathcal{F}_W^i be the kernel of h^i . Then we have the following exact sequence

$$(6.30) \quad 0 \rightarrow \mathcal{F}_W^i \rightarrow \mathcal{E}_2^i \xrightarrow{h^i} \mathcal{O}_W \rightarrow 0.$$

It is easy to see that \mathcal{F}_W^i is slope-stable and $s_{\mathcal{F}_W^i} = s_{\mathcal{E}_2^i} \times \pi^* t_W$. Recall that t_W is the section of $H^0(|L|, \mathcal{O}_{|L|}(n-2))$ vanishing at curves passing through any point in W , and moreover $\{t_W\}_W$ spans $H^0(|L|, \mathcal{O}_{|L|}(n-2))$. By Lemma 6.24 and Proposition 6.25 we have the following lemma.

Lemma 6.31. *The restriction of $\{s_{\mathcal{F}_W^i}\}_W$ to D_Θ spans $H^0(D_\Theta, \Theta^2(n)|_{D_\Theta})$ for $n \geq 2$.*

By Proposition 6.17, Lemma 6.20 and Lemma 6.31, we have the following proposition.

Proposition 6.32. *Let $X = \mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or $\widehat{\mathbb{P}}^2$. Let $c^* = (0, L, \chi = 0)$ with $L = -K_X$ or $L = -K_X + F$ with F the fiber class for X a Hirzebruch surface, and $c = (2, 0, c_2)$. Then for any polarization H on X , the strange duality map*

$$SD_{c,c^*}^g : H^0(M_H^X(c)^g, \mathcal{D}_{c,c^*}^g) \rightarrow H^0(M_H^X(c^*), \mathcal{D}_{c^*,c})$$

is surjective for all $c_2 \geq 2$.

Proof of Theorem 6.3 for cases (1) and (2). Combine Proposition 6.4 and Proposition 6.32. \square

Remark 6.33. In cases (1) and (2), there are still some conditions on the polarization H , which are required by Theorem 1.2. However, by Theorem 3.19 we know that the generating function $\sum_{d>0} \chi(M_H^X(c), \mathcal{D}_{c,c^*})$ is essentially independent of H . Hence we know that the strange duality map SD_{c,c^*}^g is an isomorphism for any polarization for $c_2 \gg 0$.

† Case (3).

In this case, we have $X = \mathbb{P}^2$ with H the hyperplane class, $c^* = (0, 2H, \chi = -1)$ and $c = (2, H, c_2 \geq 1)$. As we have shown in the proof of Proposition 6.4, the map $\pi : M_H^X(c^*) \rightarrow |2H|$ is an isomorphism, $\mathcal{D}_{c^*,c} \cong \mathcal{O}_{|2H|}(c_2 - 1)$, $\mathcal{T}_{\mathbb{P}^2}(-1)$ is slope-stable of class $(2, H, 1)$ and $s_{\mathcal{T}_{\mathbb{P}^2}(-1)}$ is nowhere vanishing.

Let W be a subset of $\{x_j^k\}_{1 \leq j \leq 6}^{1 \leq k \leq c_2-1}$ such that $\#W = c_2 - 1$. Let $\mathcal{O}_W \cong \bigoplus_{x \in W} \mathcal{O}_x$ be the structure sheaf of the subscheme W . We then choose a surjective map $h : \mathcal{T}_{\mathbb{P}^2}(-1) \rightarrow \mathcal{O}_W$ and construct a slope-stable sheaf \mathcal{F}_W^H associated to W as the kernel of h . We have

$$(6.34) \quad 0 \rightarrow \mathcal{F}_W^H \rightarrow \mathcal{T}_{\mathbb{P}^2}(-1) \xrightarrow{h} \mathcal{O}_W \rightarrow 0.$$

It is easy to see that $s_{\mathcal{F}_W^H} = s_{\mathcal{T}_{\mathbb{P}^2}(-1)} \times t_W = t_W$ and hence $\{s_{\mathcal{F}_W^H}\}_W$ spans $H^0(M(c^*), \mathcal{D}_{c^*,c})$. Hence by Proposition 6.17 we have

Lemma 6.35. *The map SD_{c,c^*} is surjective for case (3).*

Proof of Theorem 6.3 for case (3). Combine Proposition 6.4 and Lemma 6.35. \square

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